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On some growth properties related to certain type of difference polynomials

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Abstract

For an entire function f , we define the difference operators as $\Delta_c f(z) = f(z+c) - f(z)$

and $\Delta_c^n f(z) = \Delta_c^{n-1}(\Delta_c f(z))$

where c is a non-zero complex number and $n \geq 2$ being a positive integer. If $c=1$, we write $\Delta_c f(z) = \Delta f(z)$.

Now let us consider $F = f^n(f^m - 1)\prod_{j=1}^{d}(f(z+c_j))^{\gamma_j}$, where f being an entire function and n, m, $\gamma_j (j = 1,2,3,.......d)$ are all non-negative integers. Then F is called the difference monomial generated by entire f .

In this paper, we will establish some comparative growth properties of differential –difference polynomials of the above form generated by an entire function f as indicated. In fact, the results obtained here improve some earlier theorems.

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Introduction

Let f be an transcendental entire function defined in the open complex plane $\mathbb{C}$. A difference-monomial generated by f , is an expression of the form

$$F = f^n(f^m - 1)\prod_{j=1}^{d}(f(z+c_j))^{\gamma_j}$$

where m, n and γ_j are all non-negative integers.
Now for the sake of definiteness let us take,

$$M_i[f] = f^n(f^m - 1)\prod_{j=1}^{i}(f(z + c_j))^{\gamma_j}$$

where $1 \leq i \leq d$.
If $M_1[f], M_2[f], \ldots. M_n[f]$ be such monomials in f as defined above, then

$$\emptyset[f] = a_1M_1[f] + a_2M_2[f] + \cdots a_nM_n[f]$$

where $a_i \neq 0 (i = 1,2, \ldots n)$ is called a difference-polynomial generated by f.
Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum term μ(r,f) of $f = \sum_{n=0}^{\infty} a_n z^n$ on |z|=r is defined by

$$\mu(r, f) = max_{n\geq 0}(|a_n|r^n).$$

To start our paper we just recall the following definitions:
Let Ψ:[0,∞)→(0,∞) be a non-decreasing unbounded functionm, satisfying the following two conditions:

(i) $$\lim_{r\to\infty} \frac{log^{[q]}r}{log^{[q]}\Psi(r)} = 0$$

and

(ii) $$\lim_{r\to\infty} \frac{log^{[q-1]}\Psi(\alpha r)}{log^{[q]}\Psi(r)} = 1$$

for some $\alpha > 1$.

Definition 1 The Ψ-order $\rho_{(f,\Psi)}$ and Ψ-lower order $\lambda_{(f,\Psi)}$ of an entire function of an entire function f is defined as follows:

$$\rho_{(f,\Psi)} = \lim sup_{r\to\infty} \frac{log^{[2]}M(r,f)}{\log \Psi(r)} \text{ and } \lambda_{(f,\Psi)} = \lim inf_{r\to\infty} \frac{log^{[2]}M(r,f)}{\log \Psi(r)}$$

where $log^{[k]}x = \log(log^{[k-1]}x)$for k=1,2,3,.... and $log^{[0]}x = x$.
If $\rho_{(f,\Psi)} < \infty$ then $f$is of finite Ψ-order.Also $\rho_{(f,\Psi)=}0$ means that $f$is of Ψ-order zero. In this connection following Liao and Yang [6] we may give the definition as below:

Definition 2 Let f be an entire function of order zero.Then the quantities $\rho^*_{(f,\Psi)}$ and $\lambda^*_{(f,\Psi)}$ of an entire function f is defined as

$$\rho^*_{(f,\Psi)} = \lim sup_{r\to\infty} \frac{log^{[2]}M(r,f)}{\log[2]\Psi(r)} \text{ and } \lambda^*_{(f,\Psi)} = \lim inf_{r\to\infty} \frac{log^{[2]}M(r,f)}{\log[2]\Psi(r)}$$

In the line of Datta and Biswas [3] gave an alternative definition of zero Ψ-order and zero Ψ-lower order of an entire function may be given as:

Definition 3 Let f be an entire function of order zero.Then the quantities and $\rho^{**}_{(f,\Psi)}$ and $\lambda^{**}_{(f,\Psi)}$ of an entire function f is defined by:

$$\rho^{**}_{(f,\Psi)} = \lim sup_{r\to\infty} \frac{logM(r,f)}{\log \Psi(r)} \text{ and } \lambda^{**}_{(f,\Psi)} = \lim inf_{r\to\infty} \frac{logM(r,f)}{\log \Psi(r)}.$$

Since for $0 \leq r \leq R$,

$$\mu(r, f) \leq M(r, f) \leq \frac{R}{R-r}\mu(R, f)$$

it is easy to see that

$$\rho^{**}_{(f,\Psi)} = \lim sup_{r\to\infty} \frac{\log^{[2]}\mu(r,f)}{\log \Psi(r)} \text{ and } \lambda^{**}_{(f,\Psi)} = \lim inf_{r\to\infty}. \frac{log^{[2]}\mu(r,f)}{\log \Psi(r)}$$

$$\rho^*_{(f,\Psi)} = \lim sup_{r\to\infty} \frac{\log^{[2]}M(r,f)}{\log^{[2]}\Psi(r)} \text{ and } \lambda^*_{(f,\Psi)} = \lim inf_{r\to\infty}. \frac{\log^{[2]}\mu(r,f)}{\log^{[2]}\Psi(r)}$$

$$\rho^{**}_{(f,\Psi)} = \lim sup_{r\to\infty} \frac{log\mu(r,f)}{\log \Psi(r)} \text{ and } \lambda^{**}_{(f,\Psi)} = \lim inf_{r\to\infty} \frac{log\mu(r,f)}{\log \Psi(r)}.$$

Definition 4 The Ψ-type $\sigma_{(f,\Psi)}$ and Ψ-lower type $\bar{\sigma}_{(f,\Psi)}$ of an entire function f are defined as
$\sigma_{(f,\Psi)} = \limsup_{r\to\infty} \frac{\log M(r,f)}{\Psi(r)^{\rho(f,\Psi)}}$ and $\bar{\sigma}_{(f,\Psi)} = \liminf_{r\to\infty} \frac{\log M(r,f)}{\Psi(r)^{\rho(f,\Psi)}}$ $0<\rho_{(f,\Psi)} < \infty$.

With the help of notion of maximum terms of entire functions, Definition 4 can be alternatively stated as follows:

Definition 5 The Ψ-type $\sigma_{(f,\Psi)}$ and Ψ- lower type $\bar{\sigma}_{(f,\Psi)}$ of an entire function f are defined as

$$\sigma_{(f,\Psi)} = \limsup_{r\to\infty} \frac{\log\mu(r,f)}{\Psi(r)^{\rho(f,\Psi)}} \text{ and } \bar{\sigma}_{(f,\Psi)} = \liminf_{r\to\infty} \frac{\log\mu(r,f)}{\Psi(r)^{\rho(f,\Psi)}}, \; 0<\rho_{(f,\Psi)} < \infty.$$

Definition 6 A function $\lambda_{f(r)}$ is called a lower proximate order of f relative to T(r,f) if

(i) $\lambda_{f(r)}$ is non-negative and continuous for $r \geq r_0$, say,
(ii) $\lambda_{f(r)}$ is differentiable for $r \geq r_0$ except possibly at isolated points at which $\lambda'_f(r-0)$ and $\lambda'_f(r+0)$ exist,
(iii) $\lim_{r\to\infty} \lambda_{f(r)} = \lambda_f, <\infty$
(iv) $\lim_{r\to\infty} r\lambda'_{f(r)}\log r = 0$ and
(v) $\limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_{f(r)}}} = 1.$

Definition 7 A function $\rho_{f(r)}$ is called a proximate order of f relative to T(r,f) if

(i) $\rho_{f(r)}$ is non-negative and continuous for $r \geq r_0$, say,
(ii) $\rho_{f(r)}$ is differentiable for $r \geq r_0$ except possibly at isolated points at which $\rho'_f(r-0)$ and $\rho'_f(r+0)$ exist,
(iii) $\lim_{r\to\infty} \rho_{f(r)} = \rho_f, <\infty$
(iv) $\lim_{r\to\infty} r\rho'_{f(r)}\log r = 0$ and
(v) $\limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_{f(r)}}} = 1.$

In this paper we study of some aspects on the compartative growths of maximum terms of two entire functions with their corresponding left and right factors. We do not explain the standard notations and definitions on the theory of entire functions because those are available in [8] .

Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 Let f and g be any two entire functions with g(0)=0. Then for all sufficiently large values of r,

$$\mu(r, f\circ g) \geq \frac{1}{4}\mu\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right) - |g(0)|, f\right).$$

Lemma 2 Let f and g be any two entire functions. Then for every α and $0 < r < R$,

$$\mu(r, f\circ g) \leq \frac{\alpha}{\alpha-1}\mu\left(\frac{\alpha R}{R-r}\mu(R,g), f\right).$$

.

Lemma 3 Let f be a meromorphic function and g be trascendental entire.If $\rho_{f\circ g} < \infty$ then $\rho_{(f,\Psi)} = 0$.

Lemma 4 If f and g be two entire functions .Then for sufficiently large values of r ,

$$M\left(\frac{1}{8}M\left(\frac{r}{2}, g\right) - |g(0)|, f\right) \leq M(r, f\circ g) < M(M(r,g), f).$$

Lemma 5 If f be any entire function of order zero.Then

(i) $\rho^*_{(f,\Psi)} = 1$ and (ii) $\lambda^*_{(f,\Psi)} = 1$.

Lemma 6 If f be an entire function. Then for δ>0 the function $r^{\rho_f+\delta-\rho_f(r)}$ is an increasing function of r.

Lemma 7 If f be an entire function . Then for δ>0 the function $r^{\lambda_f+\delta-\lambda_f(r)}$ is an increasing function of r.

Proof: Since

$$\frac{d}{dr} r^{\lambda_f+\delta-\lambda_f(r)} = \{r^{\lambda_f+\delta-\lambda_f(r)} - r\lambda_f'(r)logr\} r^{\lambda_f+\delta-\lambda_f(r)} > 0$$

the function $r^{\lambda_f+\delta-\lambda_f(r)}$ is an increasing function of r.

Lemma 8 If f be an entire function . Then for δ>0 the function $r^{\rho_f+\delta-\rho_f(r)}$ is an increasing function of r.
Lemma 9 Let f be an entire function and

$$F = f^n(f^m-1)\prod_{j=1}^{d}(f(z+c_j))^{\gamma_j}$$

then,

$$T(r,F) = (n+m+\gamma)T(r,f) + S(r,f).$$

where

$$\gamma = \sum_{j=1}^{d}\gamma_j.$$

i.e., in other words as $\lim_{r\to\infty}\frac{S(r,f)}{T(r,f)} = 0$,
then,

$$\lim_{r\to\infty}\frac{T(r,F)}{T(r,f)} = (n+m+\gamma).$$

Analogusly, for $\emptyset[f]$, we may have the following lemma:

Lemma 10 Let f be an entite function and F, Ψ be as defined earlier. Then

$$\lim_{r\to\infty}\frac{T(r,\emptyset[f])}{T(r,f)} = (n+m+\gamma).$$

Lemma 11 Let f be an entite function and F, Ψ be as defined earlier. Then the order (lower order) of f and F are equal. Further their types $\sigma_F = (n+m+\gamma)\sigma_f$ and $\overline{\sigma_F}=(n+m+\gamma)\overline{\sigma_f}$.

Proof: In view of Lemma 8, we obtain that

$$\rho_F = \lim sup_{r\to\infty}\frac{\log T(r,F)}{logr}$$
$$=\lim sup_{r\to\infty}\frac{\log (n+m+\gamma)T(r,f)+S(r,f)}{logr}$$
$$=\lim sup_{r\to\infty}\frac{\log T(r,f)+O(1)}{logr}$$
$$= \rho_f.$$

In a like manner,

$$\rho_{(F,\Psi)} = \lim sup_{r\to\infty}\frac{\log T(r,F)}{log\Psi(r)}$$
$$=\lim sup_{r\to\infty}\frac{\log (n+m+\gamma)T(r,f)+S(r,f)}{log\Psi(r)}$$
$$=\lim sup_{r\to\infty}\frac{\log T(r,f)+O(1)}{log\Psi(r)}$$
$$= \rho_{(f,\Psi)}.$$

In the line of Lemma 8 and Lemma 9, we may obtain the following two lemmas respectively.

Lemma 12 Let f be an entire function and $F, \varphi[F]$ be defined as above. then, $\rho_F = \rho_{\emptyset[f]}$ and $\sigma_F = \sigma_{\emptyset[f]}$
The proofs are omitted.
Analogously, the following lemma can be derived:

Lemma 13 Let f be an entite function and F, Ψ be as defined earlier. Then

$\rho_{\emptyset[f]} = \rho_{(\emptyset[f],\Psi)}$,
$\lambda_{\emptyset[f]} = \lambda_{(\emptyset[f],\Psi)}$ and
$\sigma_{\emptyset[f]} = \sigma_{(\emptyset[f],\Psi)}$.

In the line of Lemma 10 we may prove the following Lemma:

Lemma 14 Let f, F and $\emptyset[f]$ be defined as above. Then

$\bar{\rho}_{(\emptyset[f],\Psi)} = \bar{\rho}_{(f,\Psi)}$,
$\bar{\lambda}_{(\emptyset[f],\Psi)} = \bar{\lambda}_{(f,\Psi)}$ and
$\bar{\sigma}_{(\emptyset[f],\Psi)} = \bar{\sigma}_{(f,\Psi)}$.

Main Results.

In this section we present the main results of the paper.

Theorem 1 Let f and g be any two entire functions with $0 < \lambda_{(f,\Psi)} \leq \rho_{(f,\Psi)} < \infty$ and $0 < \rho_{(g,\Psi)} < \infty$. Also let $0 < \bar{\sigma}_{(g,\Psi)} \leq \sigma_{(g,\Psi)} < \infty$.

Further suppose that $\emptyset[g]$ be the difference monomial in g. for $n, m, \gamma_j \geq 1$. Then

$$\limsup_{r\to\infty} \frac{log^{[2]}\mu(r, f\circ g)}{log\mu\,(r,\emptyset[g])} \geq \left(\frac{1}{n+m+\gamma}\right)\frac{\bar{\sigma}_{(g,\Psi)}\rho_{(f,\Psi)}}{\sigma_{(g,\Psi)}4^{\rho_{(g,\Psi)}}}$$

Proof: We obtain from Lemma 1 for a sequence of values of r tending to infinity,

$$log^{[2]}\,\mu(r, f\circ g) \geq log^{[2]}\left\{\frac{1}{4}\mu\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right) - |g(0)|, f\right)\right\}$$

i.e., $log^{[2]}\,\mu(r, f\circ g) \geq \left(\rho_{(f,\Psi)} - \epsilon\right)\log\left\{\frac{1}{8}\mu\left(\frac{r}{4}, g\right)\right\} + O(1)$

i.e., $log^{[2]}\,\mu(r, f\circ g) \geq \left(\rho_{(f,\Psi)} - \epsilon\right)log\frac{1}{8} + \left(\rho_{(f,\Psi)} - \epsilon\right)\log\mu\left(\frac{r}{4}, g\right) + O(1)$

... (1)

Again from the definition of lower type, we have for arbitrary positive ε and for all sufficiently large values of r, by Lemma 10

$$\log\mu\left(\frac{r}{4}, g\right) \geq \left(\bar{\sigma}_{(g,\Psi)} - \varepsilon\right)\left(\frac{\Psi(r)}{4}\right)^{\rho_{(g,\Psi)}}.$$

... (2)

Therefore from (1) and (2) it follows for a sequence of values of r tending to infinity,

$$log^{[2]}\,\mu(r, f\circ g) \geq \left(\rho_{(f,\Psi)} - \varepsilon\right)log\frac{1}{8} + \left(\rho_{(f,\Psi)} - \epsilon\right)\left(\bar{\sigma}_{(g,\Psi)} - \varepsilon\right)\left(\frac{\Psi(r)}{4}\right)^{\rho_{(g,\Psi)}} + O(1)$$

... (3)

where we choose $\varepsilon(>0)$ in such a way that

$$0 < \varepsilon < \min\{\rho_{(f,\Psi)}, \bar{\sigma}_{(g,\Psi)}\}.$$

Also for all sufficiently large values of r,

$$log\mu(r, \emptyset[g]) \leq (n+m+\gamma)\left(\sigma_{(g,\Psi)} + \varepsilon\right)(\Psi(r)^{\rho_{\emptyset[g],\Psi}})$$

$$log\mu(r, \emptyset[g]) \leq (n+m+\gamma)\left(\sigma_{(g,\Psi)} + \varepsilon\right)(\Psi(r))^{\rho_{(g,\Psi)}}.$$

... (4)

Now from (3) and (4) it follows for a sequence of values of r tending to infinity,

$$\frac{log^{[2]}\mu(r,f\circ g)}{log\mu(r,\emptyset[g])} \geq \frac{(\rho_{(f,\Psi)}-\varepsilon)log\frac{1}{8}+(\rho_{(f,\Psi)}-\varepsilon)(\bar{\sigma}_{(g,\Psi)}-\varepsilon)\left(\frac{\Psi(r)}{4}\right)^{\rho(g,\Psi)}+O(1)}{(n+m+\gamma)(\sigma_{(g,\Psi)}+\varepsilon)r^{\rho(g,\Psi)}}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty}\frac{log^{[2]}\mu(r,f\circ g)}{log\mu(r,\emptyset[g])} \geq \frac{\bar{\sigma}_{(g,\Psi)}\rho_{(f,\Psi)}}{(n+m+\gamma)\,\sigma_{(g,\Psi)}4^{\rho(g,\Psi)}}.$$

... (5)

This proves the theorem .

In this line ,Theorem 1 one may easily prove the following corollary.

Corollary 1 Let f and g be any two entire functions with $0<\lambda_{(f,\Psi)}\leq\rho_{(f,\Psi)}<\infty$ and $0<\rho_{(g,\Psi)}<\infty$. Also let $0<\sigma_{(g,\Psi)}<\infty$.

Further suppose that $\emptyset[g]$ be the difference monomial in g for $n,m,\gamma_j\geq 1$. Then

$$\limsup_{r\to\infty}\frac{log^{[2]}\mu(r,f\circ g)}{log\mu(r,\emptyset[g])}\geq\frac{\lambda_{(f,\Psi)}}{4^{\rho(g,\Psi)}}.$$

If f be any entire function of order zero then the following corollary can also be proved with the help of the growth indicator $\lambda^{**}{}_{(f,\Psi)}$ for entire f in terms of its maximum term.

Corollary 2 Let f be any entire function of order zero such that $0<\lambda^{**}{}_{(f,\Psi)}<\infty$and g be any entire function of finite order $0<\sigma_{(g,\Psi)}<\infty$. .

Further suppose that $\emptyset[g]$ be the difference monomial in g. Then

$$\limsup_{r\to\infty}\frac{log^{[2]}\mu(r,f\circ g)}{log\mu(r,\emptyset[g])}\geq\frac{\lambda^{**}{}_{(f,\Psi)}}{4^{\rho(g,\Psi)}}.$$

Remark 1 If we take $0<\lambda_{(g,\Psi)}\leq\rho_{(g,\Psi)}<\infty$ instead of " finite order with " $0<\sigma_{(g,\Psi)}<\infty$. in corollary 2 and the other conditions remain the same then with the help of growth indicators $\rho^{*}{}_{(f,\Psi)}$ for entire f in terms of its maximum terms and in view of Lemma 6 and Lemma 10 it can be carried out that

$$\limsup_{r\to\infty}\frac{log^{[2]}\mu(r,f\circ g)}{log\mu(r,\emptyset[g])}\geq 1.$$

Theorem 2 Let f and g be any two entire functions with $0<\lambda_{(f,\Psi)}\leq\rho_{(f,\Psi)}<\infty$ and $0<\rho_{(g,\Psi)}<\infty$. Also let $0<\bar{\sigma}_{(g,\Psi)}\leq\sigma_{(g,\Psi)}<\infty$.

Further suppose that $\emptyset[g]$ be the difference monomial in g. for $n,m,\gamma_j\geq 1$. Then

$$\limsup_{r\to\infty}\frac{log^{[2]}\mu(r,f\circ g)}{log\mu(r,\emptyset[g])}\leq\frac{\rho_{(f,\Psi)}\sigma_{(g,\Psi)}}{(n+m+\gamma)\bar{\sigma}_{(g,\Psi)}}.$$

Proof: Since for $0\leq r<R$,

$$\mu(r,f)\leq M(r,f)\leq\frac{R}{R-r}\mu(r,f),$$

by Lemma 4 it follows for all sufficiently large values of r that

$$\mu(r,f\circ g)\ \leq\ M(r,f\circ g)\leq M(M(r,g),f)$$
$$log^{[2]}\mu(r,f\circ g)\leq log^{[2]}M(M(r,g),f)$$
i.e., $log^{[2]}\mu(r,f\circ g)\leq(\rho_{(f,\Psi)}+\varepsilon)logM(r,g)$.

... (6)

Also for all sufficiently large values of r that

$$logM(r,g) \leq (\sigma_{(g,\Psi)} + \varepsilon)(\Psi(r))^{\rho(G,\Psi)}$$

i.e., $logM(r,g) \leq (\sigma_{(g,\Psi)} + \epsilon)(\Psi(r))^{\rho(g,\Psi)}$,by Lemma 10.

... (7)

Therefore from (6) and (7), we have for all sufficiently large values of r,

$$log^{[2]} \mu(r, f \circ g) \leq (\rho_{(f,\Psi)} + \varepsilon)(\sigma_{(g,\Psi)} + \varepsilon)(\Psi(r))^{\rho(g,\Psi)}.$$

... (8)

Again for all sufficiently large values of r,

$$log\mu(r,\emptyset[g]) \geq (n+m+\gamma)\overline{(\sigma_{(g,\Psi)}} - \varepsilon)\big(\Psi(r)\big)^{\rho(G,\Psi)}$$

i.e., $log\mu(r,\emptyset[g]) \geq (n+m+\gamma)(\sigma_{(g,\Psi)} - \varepsilon)\big(\Psi(r)\big)^{\rho(g,\Psi)}$, by Lemma 10.

... (9)

Now from (8) and (9), we obtain for all sufficiently large values of r ,

$$\frac{log^{[2]}\mu(r, f \circ g)}{log\mu(r,\emptyset[g])} \leq \frac{(\rho_{(f,\Psi)} + \varepsilon)(\sigma_{(g,\Psi)} + \varepsilon)(\Psi(r))^{\rho(g,\Psi)}}{(n+m+\gamma)(\sigma_{(g,\Psi)} - \varepsilon)\big(\Psi(r)\big)^{\rho(g,\Psi)}}$$

i.e.,

$$\limsup_{r\to\infty} \frac{log^{[2]}\mu(r,f\circ g)}{log\mu(r,\emptyset[g])} \leq \frac{\rho(f,\Psi)\sigma(g,\Psi)}{(n+m+\gamma)\bar{\sigma}_{(g,\Psi)}}.$$

This completes the proof of the theorem.

The following theorem is a natural consequence of Theorem 1 and Theorem 2.

Theorem 3 Let f and g be any two entire functions with $0 < \lambda_{(f,\Psi)} \leq \rho_{(f,\Psi)} < \infty$ and $0 < \rho_{(g,\Psi)} < \infty$. Also let $0 < \bar{\sigma}_{(g,\Psi)} \leq \sigma_{(g,\Psi)} < \infty$.

Further suppose that $\emptyset[g]$ be the difference monomial in g. for $n, m, \gamma_j \geq 1$. Then

$$\left(\frac{1}{n+m+\gamma}\right)\frac{\bar{\sigma}(g,\Psi)\rho(f,\Psi)}{\sigma_{(g,\Psi)}4^{\rho(g,\Psi)}} \leq \limsup_{r\to\infty} \frac{log^{[2]}\mu(r,f\circ g)}{log\mu(r,\emptyset[g])} \leq \frac{\rho(f,\Psi)\sigma(g,\Psi)}{(n+m+\gamma)\bar{\sigma}_{(g,\Psi)}}.$$

The proof is omitted.

If f be any entire function of order zero, then the following theorem can be carried out in the line of Theorem 1 and Theorem 2.

Theorem 4 Let f and g be any two entire functions with $0 < \lambda_{(f,\Psi)} \leq \rho_{(f,\Psi)} < \infty$ and $0 < \rho_{(g,\Psi)} < \infty$. Also let $0 < \bar{\sigma}_{(g,\Psi)} \leq \sigma_{(g,\Psi)} < \infty$. and for $n \geq 1$, and $\emptyset[g]$ is a difference polynomial in g . Then,

$$\left(\frac{1}{n+m+\gamma}\right)\frac{\bar{\sigma}(g,\Psi)\rho^{**}(f,\Psi)}{\sigma_{(g,\Psi)}4^{\rho(g,\Psi)}} \leq \limsup_{r\to\infty} \frac{log^{[2]}\mu(r,f\circ g)}{log\mu(r,\emptyset[g])} \leq \frac{\rho^{**}(f,\Psi)\sigma(g,\Psi)}{(n+m+\gamma)\bar{\sigma}_{(g,\Psi)}}.$$

Remark 2 If we take $0 < \lambda_{(g,\Psi)} \leq \rho_{(g,\Psi)}$ instead of " finite order with " $0 < \bar{\sigma}_{(g,\Psi)} \leq \sigma_{(g,\Psi)} < \infty$.in Theorem 4 and the other conditions remain the same then with the help of growth indicators $\rho^*{}_f$ for entire f in terms of its maximum term and Lemma 5 and Lemma 10 it can be carried out that

$$\frac{\lambda_{(g,\Psi)}}{\rho_{(g,\Psi)}} \leq \limsup_{r\to\infty} \frac{log^{[2]}\mu(r, f \circ g)}{log^{[2]}\mu(r,\emptyset[g])} \leq \frac{\rho_{(g,\Psi)}}{\lambda_{(g,\Psi)}}.$$

where, $\emptyset[g]$ is a difference polynomial in g .

Theorem 5 Let f and g be any two entire functions with $0 < \lambda_{(f,\Psi)} \leq \rho_{(f,\Psi)} < \infty$ and $0 < \rho_{(g,\Psi)} < \infty$. Also let $0 < \bar{\sigma}_{(g,\Psi)} \leq \sigma_{(g,\Psi)} < \infty$.and for $n \geq 1$, and $\emptyset[g]$ is a difference polynomial in g .
Then

$$\liminf_{r\to\infty} \frac{\log^{[2]}\mu(r,f\circ g)}{\log\mu(r,\emptyset[g])} \geq \frac{\lambda_{(f,\Psi)}\bar{\sigma}_{(g,\Psi)}}{(n+m+\gamma)\sigma_{(g,\Psi)}}.$$

Proof: We obtain from Lemma 1 for all sufficiently large values of r that

$$\log^{[2]}\mu(r,f\circ g) \geq \log^{[2]}\left\{\frac{1}{4}\mu\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right),f\right)\right\}$$

i.e., $\log^{[2]}\mu(r,f\circ g) \geq (\lambda_{(f,\Psi)}-\varepsilon)\log\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)\right)+O(1)$

i.e., $\log^{[2]}\mu(r,f\circ g) \geq (\lambda_{(f,\Psi)}-\varepsilon)\log\frac{1}{8}+(\lambda_{(f,\Psi)}-\varepsilon)\log\mu\left(\frac{r}{4},g\right)+O(1).$

... (10)

Therefore from (2) and (10), it follows for all sufficiently large vaules of r ,

$\log^{[2]}\mu(r,f\circ g) \geq (\lambda_{(f,\Psi)}-\varepsilon)\log\frac{1}{8}+(\lambda_{(f,\Psi)}-\varepsilon)(\overline{\sigma}_{(g,\Psi)}-\epsilon)\frac{(\Psi(r))^{\rho(g,\Psi)}}{4}+O(1).$

... (11)

Combining (4) and (11), we obtain for all sufficiently large values of r ,

$$\frac{\log^{[2]}\mu(r,f\circ g)}{\log\mu(r,\emptyset[g])} \geq \frac{\mu(r,f\circ g)\geq(\lambda_{(f,\Psi)}-\varepsilon)\log\frac{1}{8}+(\lambda_{(f,\Psi)}-\varepsilon)(\overline{\sigma}_{(g,\Psi)}-\epsilon)\frac{(\Psi(r))^{\rho(g,\Psi)}}{4}+O(1)}{(n+m+\gamma)(\sigma_{(g,\Psi)}+\varepsilon)(\Psi(r))^{\rho(g,\Psi)}}.$$

Since $\varepsilon(>0)$ is arbitrary it follows from the above that

$$\liminf_{r\to\infty} \frac{\log^{[2]}\mu(r,f\circ g)}{\log\mu(r,\emptyset[g])} \geq \frac{\lambda_{(f,\Psi)}\bar{\sigma}_{(g,\Psi)}}{(n+m+\gamma)\sigma_{(g,\Psi)}}.$$

Thus the theorem follows.

Theorem 6 Let f and g be any two entire functions with $0<\lambda_{(f,\Psi)}\leq\rho_{(f,\Psi)}<\infty$ and $0<\rho_{(g,\Psi)}<\infty$. Also let $0<\bar{\sigma}_{(g,\Psi)}\leq\sigma_{(g,\Psi)}<\infty$.and for $n\geq 1$, and $\emptyset[g]$ is a difference polynomial in g
Then

$$\liminf_{r\to\infty} \frac{\log^{[2]}\mu(r,f\circ g)}{\log\mu(r,\emptyset[g])} \leq \frac{\lambda_{(f,\Psi)}\sigma_{(g,\Psi)}}{(n+m+\gamma)\bar{\sigma}_{(g,\Psi)}}.$$

.

.**Proof:** Since for $0\leq r<R$,

$$\mu(r,f)\leq M(r,f)\leq\frac{R}{R-r}\mu(R,f) ,$$

by Lemma 4 and the above inequality it follows for a sequence of values of r tending to infinity,

$\mu(r,f\circ g) \leq M(r,f\circ g)\leq M(M(r,g),f)$

i.e., $\log^{[2]}\mu(r,f\circ g) \leq \log^{[2]}M(M(r,g),f)$

i.e., $\log^{[2]}\mu(r,f\circ g) \leq (\lambda_{(f,\Psi)}+\varepsilon)\log M(r,g).$

... (12)

Therefore from (7) and (12), we have for a sequence of values of r tending to infinity,

$$\log^{[2]}\mu(r,f\circ g) \leq (\lambda_{(f,\Psi)}+\varepsilon)(\sigma_{(g,\Psi)}+\varepsilon)(\Psi(r))^{\rho(g,\Psi)}$$

... (13)

Now from (9) and (13), we have for a sequence of values of r tending to infinity, and by Lemma 10,

$$\frac{\log^{[2]}\mu(r,f\circ g)}{\log\mu(r,\emptyset[g])} \le \frac{(\lambda_{(f,\Psi)}+\varepsilon)(\sigma_{(g,\Psi)}+\varepsilon)(\Psi(r))^{\rho(g,\Psi)}}{(n+m+\gamma)(\sigma_{(g,\Psi)}-\varepsilon)(\Psi(r))^{\rho(g,\Psi)}}$$

i.e.,

$\liminf_{r\to\infty} \frac{\log^{[2]}\mu(r,f\circ g)}{\log\mu(r,\emptyset[g])} \le \frac{\lambda_{(f,\Psi)}\sigma_{(g,\Psi)}}{(n+m+\gamma)\bar{\sigma}_{(g,\Psi)}}$

This completes the proof.

The following theorem is a natural consequence of Theorem 5 and Theorem 6.

Theorem 7 Let f and g be any two entire functions with $0<\lambda_{(f,\Psi)}\le\rho_{(f,\Psi)}<\infty$ and $0<\rho_{(g,\Psi)}<\infty$. Also let $0<\bar{\sigma}_{(g,\Psi)}\le\sigma_{(g,\Psi)}<\infty$.and for $n\ge 1$, and $\emptyset[g]$ is a difference polynomial in g

Then

$$\frac{\lambda_{(f,\Psi)}\bar{\sigma}_{(g,\Psi)}}{(n+m+\gamma)\sigma_{(g,\Psi)}} \le \liminf_{r\to\infty} \frac{\log^{[2]}\mu(r,f\circ g)}{\log\mu(r,\emptyset[g])} \le \frac{\lambda_{(f,\Psi)}\sigma_{(g,\Psi)}}{(n+m+\gamma)\bar{\sigma}_{(g,\Psi)}}.$$

The proof is omitted.

If f be any entire function of order zero, then the following theorem can also be carried out in the line of Theorem 5 and Theorem 6.

Theorem 8 Let f and g be any two entire functions with $0<\lambda_{(f,\Psi)}\le\rho_{(f,\Psi)}<\infty$ and $0<\rho_{(g,\Psi)}<\infty$. Also let $0<\bar{\sigma}_{(g,\Psi)}\le\sigma_{(g,\Psi)}<\infty$.and for $n\ge 1$, and $\emptyset[g]$ is a difference polynomial in g

Then

$\frac{\lambda^{**}{(f,\Psi)}\bar{\sigma}_{(g,\Psi)}}{(n+m+\gamma)\sigma_{(g,\Psi)}} \le \liminf_{r\to\infty} \frac{\log^{[2]}\mu(r,f\circ g)}{\log\mu(r,\emptyset[g])} \le \frac{\lambda^{**}{(f,\Psi)}\sigma_{(g,\Psi)}}{(n+m+\gamma)\bar{\sigma}_{(g,\Psi)}}.$

Remark 3 If we take $0<\lambda_{(g,\Psi)}\le\rho_{(g,\Psi)}<\infty$ instead of " finite order" with $0<\bar{\sigma}_{(g,\Psi)}\le\sigma_{(g,\Psi)}<\infty$ in Theorem 8 and the other conditions remain the same then with the help of growth indicators $\rho *_{(f,\Psi)}$ for entire f in terms of its maximum terms and in view of Lemma 5 and Lemma 10 it can be carried out that

$\frac{\lambda_{(g,\Psi)}}{\rho_{(g,\Psi)}} \le \liminf_{r\to\infty} \frac{\log^{[2]}\mu(r,f\circ g)}{\log^{[2]}\mu(r,\emptyset[g])} \le \frac{\rho_{(g,\Psi)}}{\lambda_{(g,\Psi)}},$

where $\emptyset[g]$ is a difference polynomial in g .

In this line of Theorem 6 one may easily prove the following two corollaries:

Corollary 3 Let f and g be any two entire functions with $0<\lambda_{(f,\Psi)}\le\rho_{(f,\Psi)}<\infty$ and $0<\rho_{(g,\Psi)}<\infty$. Also let $0<\sigma_{(g,\Psi)}\le\bar{\sigma}_{(g,\Psi)}<\infty$ and for $n\ge 1$, and $\emptyset[g]$ is a difference polynomial in g

Then $\liminf_{r\to\infty} \frac{\log^{[2]}\mu(r,f\circ g)}{\log\mu(r,\emptyset[g])} \le \rho_{(f,\Psi)}$.

Corollary 4 Let f be an entire function of order zero such that $0<\lambda^{**}{}_{(f,\Psi)}<\infty$ and g be any entire function of finite order with $0<\sigma_{(g,\Psi)}<\infty$.

Then

$$\liminf_{r\to\infty} \frac{\log^{[2]}\mu(r,f\circ g)}{\log\mu(r,\emptyset[g])} \le \rho **_{(f,\Psi)}$$

Theorem 9 Let f be an entire function of order zero and g be an entire such that $\rho_{(g,\Psi)}$ is finite. Also let $\emptyset[g]$ is a difference mononomial in g .Then for $0\le r<R$,

$\liminf_{r\to\infty} \frac{\log^{[2]}\mu(r,f\circ g)}{\log\mu(r,\emptyset[g])} \le 3.\rho^{**}{}_{f}.2^{\rho_g}.$

Proof: If, $\rho^{**}{}_{f}=\infty$then the results is obvious. So we suppose that $\rho^{**}{}_{f}<\infty$

Since for $0\le r<R$,

$$\mu(r,f) \le M(r,f) \le \frac{R}{R-r}\mu(R,f) \ ,$$

by Lemma 4 for $\varepsilon(>0)$ and for all sufficiently large values of r ,

$$log\mu(r,f\circ g) \le (\rho^{**}{}_f + \varepsilon)logM(r,g).$$

Since $T(r,g) \le log^{+}M(r,g)$ and $\varepsilon(>0)$ is arbitrary, it follows from the above that

$$\liminf_{r\to\infty} \frac{log^{[2]}\mu(r,f\circ g)}{log\mu(R,\emptyset[g])+O(1)} \le \rho^{**}{}_f \liminf_{r\to\infty} \frac{logM(r,g)}{logT(r,g)}$$

i.e., $\liminf_{r\to\infty} \frac{log^{[2]}\mu(r,f\circ g)}{log\mu(R,\emptyset[g])+O(1)} \le \rho^{**}{}_f \liminf_{r\to\infty} \frac{logM(r,g)}{T(r,g)}$

... (14)

Since

$$\limsup_{r\to\infty} \frac{T(r,g)}{r^{\rho_g(r)}} = 1,$$

for given $\varepsilon(0<\varepsilon<1)$ we get for all sufficiently large values of r,

$$T(r,g) < (1+\varepsilon)\, r^{\rho_g(r)}$$

... (15)

and for a sequence of values of r tending to infinity ,

$$T(r,g) > (1-\varepsilon)\, r^{\rho_g(r)}$$

... (16)

Since $logM(r,g) \le 3T(2r,g)$,

for a sequence of values of r tending to infinity we get for any $\delta(>0)$

$$\frac{logM(r,g)}{T(r,g)} \le \frac{3(1+\varepsilon)}{(1-\varepsilon)} \cdot \frac{(2r)^{\rho_\varepsilon+\delta}}{(2r)^{\rho_\varepsilon+\delta+\rho_g(2r)}} \cdot \frac{1}{r^{\rho_g(r)}}$$

$$\le \frac{3(1+\varepsilon)}{(1-\varepsilon)}(2)^{\rho_\varepsilon+\delta},$$

because $r^{\rho_\varepsilon+\delta+\rho_g(r)}$ is increasing function of r . Since $\varepsilon(>0)$ and $\delta(>0)$ are arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{logM(r,g)}{T(r,g)} \le 3.(2)^{\rho_\varepsilon}$$

... (17)

Therefore from (14) and (17) it follows that

$$\liminf_{r\to\infty} \frac{log^{[2]}\mu(r,f\circ g)}{log\mu(r,\emptyset[g])} \le 3.\rho **_f.2^{\rho_g}.$$

Thus the theorem is established.

In the line of Theorem 9 one can easily prove the following theorem using the definition of lower proximate order.

Theorem 10 Let f be an entire function of order zero and g be any entire with $\lambda_g < \infty$. Then for $0 \le r < R$,

$$\liminf_{r\to\infty} \frac{log^{[2]}\mu(r,f\circ g)}{log\mu(r,\emptyset[g])} \le 3.\rho^{**}{}_f.2^{\lambda_g},$$

where, $\emptyset[g]$ is defined earlier.

Theorem 11:Let f and g be two non constant entire functions such that f is of lower order zero and $\lambda **_f$ and λ_g are finite. Then

$$\limsup_{r\to\infty} \frac{\log\mu(R, f\circ g)}{\log\mu(\frac{r}{2}, \emptyset[g])} \geq \frac{1}{3} \cdot \frac{\lambda **_f}{4^{\lambda_g}},$$

. where $\emptyset[g]$ is defined earlier.

Proof: If $\lambda **_f = 0$ then the result is obvious. So we suppose that $\lambda^{**}_f > 0$ Since for $0 \leq r < R$,

$$\mu(r, f) \leq M(r, f) \leq \frac{R}{R-r}\mu(R, f) \ ,$$

With the help of the above inequality and Lemma 5 and for $\varepsilon(0 < \varepsilon < \min\{\lambda^{**}_f, 1\})$ we get for all sufficiently large values of r ,

$$\log\mu(R, f\circ g) + O(1) \ \geq \ \log M(r, f\circ g) \geq \log M((\tfrac{1}{8}M((\tfrac{r}{2}), g) - |g(0)|, f)$$

i.e.,. $\log\mu(R, f\circ g) + O(1) \ \geq \ (\lambda^{**}_f - \varepsilon)\log\{(\frac{1}{8})M((\frac{r}{4}), g) - |g(0)|\}$

i.e.,. $\log\mu(R, f\circ g) + O(1) \ \geq \ (\lambda^{**}_f - \varepsilon)\log\{(\frac{1}{9})M((\frac{r}{4}), g)\}$

i.e.,. $\log\mu(R, f\circ g) + O(1) \ \geq \ (\lambda^{**}_f - \varepsilon)\log M((\frac{r}{4}), g) + (\frac{1}{3})(\lambda^{**}_f - \varepsilon)\log(\frac{1}{9})$

i.e.,. $\log\mu(R, f\circ g) + O(1) \ \geq \ (\lambda^{**}_f - \varepsilon)T\left(\frac{r}{4}, g\right) + O(1)$.

... (18)

Since, $\liminf_{r\to\infty} \frac{T(r,, \emptyset[g])}{r^{\lambda_\varepsilon(r)}} = \frac{1}{n+m+\gamma}$

for given $\varepsilon(> 0)$ we get for all sufficiently large values of r ,

$$T(r, \emptyset[g]) > \frac{1}{(n+m+v)}(1-\varepsilon)r^{\lambda_\varepsilon(r)}$$

... (19)

and for a sequence of values of r tending to infinity,

$$T(r, \emptyset[g]) < \frac{1}{(n+m+v)}(1+\varepsilon)r^{\lambda_\varepsilon(r)}.$$

... (20)

From (18) and (19) we get for $\delta(> 0)$ and for all sufficiently large values of r

$$\log\mu(R, f\circ g) + O(1) \geq (\lambda^{**}_f - \varepsilon)(1-\varepsilon)\frac{\left(\frac{r}{4}\right)^{\lambda_\varepsilon+\delta}}{(n+m+\gamma)4^{\lambda_\varepsilon+\delta}}.$$

Since $r^{\lambda_\varepsilon+\delta+\lambda_g(r)}$ is an increasing function of r it follows for all sufficiently large values of r that

$$\log\mu(R, f\circ g) + O(1) \geq (\lambda^{**}_f - \varepsilon)(1-\varepsilon)\frac{r^{\lambda_\varepsilon(r)}}{(n+m+\gamma)4^{\lambda_\varepsilon+\delta}}.$$

... (21)

So by (20) and (21) we get for a sequence of values of r tending to infinity

$$\log\mu(R, f\circ g) + O(1) \ \geq \ \left(\lambda^{**}_f - \varepsilon\right)\frac{(1+\varepsilon)}{(1-\varepsilon)} \cdot \frac{(n+m+\gamma)T(r, \emptyset[g])}{(n+m+\gamma)4^{\lambda_\varepsilon+\delta}}$$

i.e., $\log\mu(R, f\circ g) + O(1) \ \geq \left(\lambda^{**}_f - \varepsilon\right)\frac{(1+\varepsilon)}{(1-\varepsilon)}\frac{T(r, \emptyset[g])}{4^{\lambda_\varepsilon+\delta}}$

i.e., $\log\mu(R, f\circ g) + O(1) \ \geq \ \left(\lambda^{**}_f - \varepsilon\right)\frac{(1+\varepsilon)}{3(1-\varepsilon)}\frac{\log M(\frac{r}{2}, \emptyset[g])}{4^{\lambda_\varepsilon+\delta}}.$

Since $\varepsilon(>0)$ and $\delta(>0)$ are arbitrary it follows from above that

$$\limsup_{r\to\infty} \frac{\log\mu(R, f\circ g)+O(1)}{\log\mu(\frac{r}{2},\emptyset[g])} \geq \frac{1}{3}.\frac{\lambda^{**}{}_f}{4^{\lambda_g}}$$

i.e.,

$$\limsup_{r\to\infty} \frac{\log\mu(R, f\circ g)}{\log\mu(\frac{r}{2},\emptyset[g])} \geq \frac{1}{3}.\frac{\lambda^{**}{}_f}{4^{\lambda_g}}$$

which is independent of n, m, γ.

Thus the theorem is proved.

Conclusion

The behaviour of the logarithmic derivative of an entire function is of much useful in the study of many problems in the value distribution theory for such functions as well as in the study of properties of the solutions of certain types of differential equations. The growth of integrated modulii of the logarithimic derivative with respect to Nevanlinnas charecteristic functions in case of meromorphic functions is an active area of research and therefore the estimation of respective growth indicators can be treated analogously. Therefore in the present paper expectation of all the treatments made on difference polynomials can be done from the view point of the interrelationship between integrated moduli of the logarithmic derivative of a meromorphic function and that of Nevanlinna's characteristic function.

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