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# STUDY ON THE CLASSIFICATION OF COPULA FUNCTIONS

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## Abstract

This article is a review article, which classifies the Copula function and studies it separately, Elliptical Copula function, Copula function of the Archimedes, Extreme value Copula function, Mixed Copula function, Time-dependent Copula functions,Variable structure Copula function.

**Key words**: Copula; function

## 1. Elliptical Copula function

Elliptic Copula function has elliptical contour distribution, among them, Gauss Copula function and t-Copula function are the most widely used, The advantage of elliptic Copula function is that it can construct Copula functions with different degrees of dependence.

**(1)** Two dimensional Gauss Copula function:

$$C(u,v;\rho)=\int_{-\infty}^{\Phi^{-1}(u)}\int_{-\infty}^{\Phi^{-1}(v)}\frac{1}{2\pi\sqrt{1-\rho^2}}\exp(-\frac{s^2-2\rho st+t^2}{2(1-\rho^2)})\mathrm{d}s\mathrm{d}t \tag{1-1}$$

$\rho$ Is a dependent parameter, and value between (-1,1), $\Phi$ and $\Phi^{-1}$ are standard normal distribution and its inverse function, respectively;

**(2)** Two dimensional t-Copula Copula function:

$$C(u,v;\rho,\ \kappa)=\int_{-\infty}^{T_\kappa^{-1}(u)}\int_{-\infty}^{T_\kappa^{-1}(v)}\frac{1}{2\pi\sqrt{1-\rho^2}}\exp(1+\frac{s^2-2\rho st+t^2}{\kappa(1-\rho^2)})^{\frac{-(\kappa+2)}{2}}\mathrm{d}s\mathrm{d}t \tag{1-2}$$

$\rho$ is a dependent parameter, and value between(-1,1), $\kappa$ is a degree of freedom, $T_\kappa$ and $T_\kappa^{-1}$ are t-distribution and its inverse function of $\kappa$.

**(3)** Multidimensional t-Copula Copula function:

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Distribution function:

$$C(u_1,\cdots,u_n,\cdots,u_N;\rho,\quad \kappa)=T_{\rho,\kappa}(t_\kappa^{-1}(u_1),\cdots,t_\kappa^{-1}(u_n),\cdots,t_\kappa^{-1}(u_N)) \tag{1-3}$$

Density function:

$$c(u_1,\cdots,u_n,\cdots,u_N;\rho,\quad \kappa)=|\rho|^{-\frac{1}{2}}\frac{\Gamma\left(\frac{\kappa+N}{2}\right)\left[\Gamma\left(\frac{\kappa}{2}\right)\right]^N}{\left[\Gamma\left(\frac{\kappa+1}{2}\right)\right]^N\Gamma\left(\frac{\kappa}{2}\right)}\frac{\left(1+\frac{1}{\kappa}\varsigma^{-1}\rho^{-1}\varsigma\right)^{-\frac{\kappa+N}{2}}}{\prod_{n=1}^{N}\left(1+\frac{\varsigma_n^2}{\kappa}\right)^{-\frac{\kappa+1}{2}}} \tag{1-4}$$

$\rho$ is a positive proof, the elements on the diagonal are 1, $|\rho|$ is the value of the matrix determinant $\rho$ ,The correlation coefficient matrix of standard multidimensional t-distribution $T_{\rho,\kappa}$ is $\rho$ , The degree of freedom is $\kappa$ , the inverse function of the t-distribution is $t_\kappa^{-1}(\cdot)$ , The degree of freedom is $\kappa$ , $\varsigma_n=t_\kappa^{-1}(u_n)$ .

## 2. Copula function of the Archimedes

Copula function of the Archimedes has many advantages, from the expression structure, we find that the Copula function of the Archimedes has symmetry and associability; Copula function of the Archimedes is easy to calculate, therefore, in most empirical analysis, Copula function of the Archimedes is usually used to build a model.

The two-dimensional Copula function of the Archimedes is defined as:

**Definition 2.1** hypothesis $\varphi: I\to[0,+\infty]$ is a convex, continuous, strictly subtraction function , that is $\sum_{n=1}^{N}\varphi(\mu_n)\le\varphi(0)$ , and to arbitrary $0\le\mu\le1$ , obtain $\varphi(1)=0$ , $\varphi'(\mu)\le0$ $\varphi''(\mu)$ $\varphi^{[-1]}$ is a quasi inverse function of $\varphi$ .

$$C(u,v)=\varphi^{[-1]}(\varphi(u)+\varphi(v)) \tag{2-1}$$

This Copula function is called the Copula function of the Archimedes cluster, function $\varphi$ is a Copula generating element (parent function). Thus, the Copula function of the Archimedes is uniquely determined by generator functions.

The n-dimensional Copula function of the Archimedes can be obtained from the expression of the two-dimensional Copula function of the Archimedes. Its expression is:

$$C(u_1,\cdots,u_n)=\varphi^{[-1]}(\varphi_1(u_1)+\cdots+\varphi(u_n)) \tag{2-2}$$

The Copula function of the Archimedes is uniquely determined by the generator function.Therefore, different Copula functions of the Archimedes correspond to different generators. So there are many Copula function of the Archimedes.

**(1)** Gumbel Copula function

$$C(u,v) = MAX((u^{-\theta} + v^{-\theta} - 1)^{\frac{-1}{\theta}}, 0) \quad (2\text{-}3)$$

The scope of $\theta$ is $[-1, \infty)/0$, generating element is $\varphi(\omega) = (-\log(\omega))^{-\theta}$.

**(2)** Clayton Copula function

$$C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)) \quad (2\text{-}4)$$

The scope of $\theta$ is $[1, \infty)$, generating element is $\varphi(\omega) = \theta^{-1}(\omega^{-1} - 1)$.

**(3)** Frank Copula function

$$C(u,v) = -\theta^{-1}\log(1 - \frac{(1-e^{-\theta u})(1-e^{-\theta v})}{(1-e^{-\theta})}) \quad (2\text{-}5)$$

The scope of $\theta$ is $\theta \neq 0$, generating element is $\varphi(\omega) = -\log\left(\frac{e^{-\theta\omega} - 1}{e^{-\theta} - 1}\right)$, the estimated value of Similarity index *Kendall's Tau* is $1 + \frac{4\{D_1(\theta) - 1\}}{\theta}$. And D is a first order Dubai (Debye) function, $D_1(\theta) = \frac{1}{\theta}\int_0^{\theta} \frac{t}{\exp(t) - 1} d_t$

**(4)** Ali-Mikhail-Haq（AMH）Copula function

$$C(u_1, \cdots, u_n) = \frac{1-\theta}{\prod_{i=1}^{n} \frac{1-\theta(1-u_i)}{u_i} - \theta} \quad (2\text{-}6)$$

The scope of $\theta$ is $-1 \prec \theta \prec 1$, generating element is $\varphi(\varpi) = In\frac{1-\theta(1-\varpi)}{\varpi}$.

**(5)** Joe Copula function

$$C(u_1, \cdots, u_n) = 1 - [1 - \prod_{i=1}^{n}(1 - (1-u_i)^{\theta})]^{1/\theta} \quad (2\text{-}7)$$

The scope of $\theta$ is $\theta \geq 1$, generating element is $\varphi(\varpi) = -In\left[1 - (1-\varpi)^{\theta}\right]$.

Gumbel function and Clayton functioneach have asymmetric structures with high tail and high tail, and Frank function is a symmetric structural function, but when the $\theta \prec 0$, the probability density is mainly concentrated on the sub diagonal line.

**3. Extreme value Copula function**

Expression in the definition of Joe (1997). If the Copula function has the following equality relation, that is the extreme value Copula function:

$$C(u_1^t,\cdots,u_n^t,\cdots,u_N^t) = C^t(u_1,\cdots,u_n,\cdots,u_N), \qquad \forall t > 0 \qquad (3\text{-}1)$$

The following conclusions can be found by examining the correlation between extreme Copula and multidimensional extreme value theory:

$$G(\chi_1^+,\cdots,\chi_n^+,\cdots,\chi_N^+) = C(G_1(\chi_1^+),\cdots,G_n(\chi_n^+),\cdots,G_N(\chi_N^+) \qquad (3\text{-}2)$$

In which G is a joint distribution of multidimensional extremum $\left(\chi_1^+,\cdots\chi_N^+\right), G_n$ is a non degenerate one-dimensional extreme value distribution, C is an extreme value Copula function.

**4. Mixed Copula function**

The rapid development of financial market, the rapid growth of financial products, the large difference between most of the financial products, can not use a single Copula function to construct the related structure between different financial products. We usually select three Copula functions of Frank, Gumbel and Clayton in Copula function of the Archimedes, According to its linear relation, the corresponding weight is assigned to construct M-Copula function. The M-Copula function relation is:

$$MC = \omega_g C_g + \omega_c C_c + \omega_f C_f \qquad (4\text{-}1)$$

$\omega_g, \omega_c, \omega_f \geq 0, \omega_g + \omega_c + \omega_f = 1$. Of the 6 parameters contained in the MC, Correlation between response variables through $(\alpha,\theta,\lambda)$, 3 correlation parameters, weighted parameter vector $(\omega_g, \omega_c, \omega_f)$ reflecting the correlation patterns between variables, the weights can fully reflect the characteristics of the related patterns.

**5. Time-dependent Copula functions**

Patton proposed a time-varying Copula model in 2001, the model changes dynamically with time, that is, the structure of the model remains unchanged, but the parameters in the model change correspondingly with time. Therefore, when establishing the time-dependent Copula model, we focus on the analysis of the time-varying characteristics of the parameters in the model, and the fitting process of the edge distribution in the model is the same as that of other Copula models.

In the commonly used time-varying correlation Copula model, the marginal distribution of the two variables of the time-dependent two normal Copula model can be arbitrarily chosen. The correlation structure between the two variables is described by a two yuan normal Copula function; the other is a time-dependent two element Joe-Clayton Copula model, and the parameter and bar in the model function There is a one-to-one correspondence between the tail correlation coefficient of the part.

**6. Variable structure Copula function**

There are three main forms of the variable structure Copula model: the edge distribution model has variable structure, the Copula function has variable structure, the edge distribution model and the Copula function all has variable structure. According to the changing forms of the variable structure Copula model, the following will introduced respectively.

**(1) The wave model of phased modeling**

The volatility model is that there is a variable structure in the edge distribution in the process of establishing the Copula model. It is necessary to cut the financial time series into multiple periods of fluctuation according to the unified criterion, and set up the fluctuation model respectively for each wave period after cutting. The fluctuation period can be cut through the subjective method and the fluctuation state of the financial time series in the event cycle is divided. The objective method, according to the statistical method of Bayes diagnosis, determines the nodes which have significant changes in the fluctuation structure in the return sequence.

**(2) Construction of Copula model in stages**

First, the probability integral transformation is carried out for multiple time series. It is similar to the partition standard in the wave model to cut the processed time series, construct the related structure Copula function with the time series of different time periods, and consider the Copula function part of the variable structure Copula model of time series $\{x_t\}$ and $\{y_t\}$.

**(3) A two-dimensional Copula model with tail variable structure characteristics**

When extreme events occur in financial markets, the internal and external correlations tend to be more closely associated with significant asymmetrical tail dependence. The variable structure model is characterized by different Copula functions, and the Clayton Copula function is more sensitive to the tail variation of the sequence distribution, and the Gumbel Copula function is more sensitive to the tail variation in the temporal distribution, and a

Copula model that can capture both the tail and the lower tails at the same time is established. For example, we choose the geometric Clayton Copula function and the Gumbel Copula function to establish the tail variable structure model. A two-dimensional Copula model with tail variable structure can be constructed for(0,1) , uniformly distributed $\{u\}_{t=1}^{T}$ and $\{v\}_{t=1}^{T}$ sequences.

$$RSC_t = D_t^{lo} C_{Cl}(u_t, v_t) + D_t^{up} C_G(u_t, v_t) + (1 - D_t^{lo} - D_t^{up}) C(u_t, v_t) \quad (6\text{-}1)$$

The two-dimensional Copula function $RSC_t$ represents the sequence correlation structure at time t. $C_{cl}(\mu_t, v_t)$, $C_G(\mu_t, v_t)$ represent Clayton Copula function and Gumbel function respectively, $C_G(\mu_t, v_t)$ is an arbitrary two-dimensional Copula function. Variables $D_t^{low}$ and $D_t^{up}$, are defined as:

$$D_t^{lo} \equiv 1\{u_t < \varepsilon, v_t < \varepsilon\}, D_t^{up} \equiv 1\{u_t > 1 - \varepsilon, v_t > 1 - \varepsilon\} \quad (6\text{-}2)$$

, $\varepsilon$ is a given threshold, $0 \prec \varepsilon \prec 0.5$.

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