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Numerical Solution of Seventh Order Boundary Value Problems by Petrov-Galerkin Method with Quintic B-splines as basis functions and Sextic B-splines as weight functions

K.N.S. Kasi Viswanadham[1]

S V Kiranmayi.Ch[2]

[1,2]Department of Mathematics
National Institute of Technology, Warangal, India

Abstract

In this paper a finite element method involving Petrov-Galerkin method with quintic B-splines as basis functions and sextic B-splines as weight functions has been developed to solve a general seventh order boundary value problem with a particular case of boundary conditions. The basis functions are redefined into a new set of basis functions which vanish on the boundary where the Dirichlet, the Neumann type of boundary conditions and the second order derivative boundary condition at the left boundary are prescribed. The weight functions are also redefined into a new set of weight functions which in number match with the number of redefined basis functions. The proposed method was applied to solve several examples of seventh order linear and nonlinear boundary value problems. The obtained numerical results were found to be in good agreement with the exact solutions available in the literature.

Keywords: Petrov-Galerkin method, Quintic B-spline, Sextic B-spline, Seventh order boundary value problem, Absolute error

1. INTRODUCTION

In this paper, we consider a general seventh order linear boundary value problem

$$\begin{aligned} &p_0(t)u^{(7)}(t)+p_1(t)u^{(6)}(t)+p_2(t)u^{(5)}(t)+p_3(t)u^{(4)}(t) \\ &+p_4(t)u'''(t)+p_5(t)u''(t)+p_6(t)u'(t)+p_7(t)u(t)=b(t), \quad c<x<d \end{aligned} \tag{1}$$

subject to boundary conditions

$$u(c)=A_0, u(d)=C_0, u'(c)=A_1, u'(d)=C_1, u''(c)=A_2, u''(d)=C_2, u'''(c)=A_3 \tag{2}$$

where A_0, C_0, A_1, C_1, A_2, C_2, A_3 are finite real constants and $p_0(t)$, $p_1(t)$, $p_2(t)$, $p_3(t)$, $p_4(t)$, $p_5(t)$, $p_6(t)$, $p_7(t)$ and $b(t)$ are all continuous functions defined on the interval $[c, d]$.

The seventh order boundary value problems generally arise in modelling induction motors with two rotor circuits. The induction motor behavior is represented by a fifth order differential equation model. This model contains two stator state variables, two rotor state variables and one shaft speed. Normally, two more variables must be added to account for the effects of a second rotor circuit representing deep bars, a starting cage or rotor distributed parameters. To avoid the computational burden of additional state variables when additional rotor circuits are required, model is often limited to the fifth order and rotor impedance is algebraically altered as function of rotor speed. This is done under the assumption that the frequency of rotor currents depends on rotor speed. This approach is efficient for the steady state response with sinusoidal voltage, but it does not hold up during the transient conditions, when rotor frequency is not a single value. The behavior of such models is shown in the seventh order boundary value problems [12]. The existence and uniqueness of the solution for these types of problems have been discussed in Agarwal [11]. Finding the analytical solutions of such type of boundary value problems in general is not possible. Over the years, many researchers have worked on seventh order boundary value problems by using different methods for numerical solutions. Siddiqi et al. [13] developed the solution of special type of seventh order boundary value problems by using Differential transformation method and they provided the solution in the form of a rapidly convergent series. Siddiqi et al. [14] presented the variational iteration principle to solve a special case of seventh order boundary value problems after transforming the given differential equation into a system of integral equations. Siddiqi and Iftikhar [15] presented Adomian decomposition method to solve the seventh order boundary value problems. Siddiqi and Iftikhar [16] provided the numerical solution of higher order boundary value problems by using Homotopy analysis method. Siddiqi and Iftikhar [17] presented variation of parameters method to solve a special case of seventh order boundary value problems and they obtained the solution in the form of a rapidly convergent series. Siddiqi and Iftikhar [18] presented variational iteration homotopy perturbation method to solve the seventh order boundary value problems, where the variational iteration homotopy perturbation method is formulated by coupling of variational iteration method and homotopy perturbation method. Siddiqi and Iftikhar [19] presented the variational iteration technique for the solution of seventh order boundary value problems by using He's polynomials. Mustafa and Ali [8], Ghazala and Rehman [3] obtained the solution of a special case of seventh order boundary value problems by using Reproducing kernel Hilbert space method and Reproducing kernel method respectively. Petrov Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions has been used to solve a general seventh order boundary value problem [6]. So far, seventh order boundary value problems have not been solved by using Petrov-Galerkin method with quintic B-splines as basis functions and sextic B-splines as weight functions. This motivated us to solve a seventh order boundary value problem by Petrov-Galerkin method with quintic B-splines as basis functions and sextic B-splines as weight functions.

In this paper, we try to present a simple finite element method which involves Petrov-Galerkin approach with quintic B-splines as basis functions and sextic B-splines as weight functions to solve a general seventh order boundary value problem of the type (1)-(2). This paper is organized as follows. Section 2 deals with the justification for using Petrov-Galerkin Method. In Section 3, a description of Petrov-Galerkin method with quintic B-splines as basis functions and sextic B-splines as weight functions is explained. In particular we first introduce the concept of quintic B-splines, sextic B-splines and followed by the proposed method to solve the boundary value problem of the type (1) and (2). In Section 4, the procedure to solve the nodal parameters has been presented. In section 5, the proposed method is tested on several linear and nonlinear boundary value problems. The solution to a nonlinear problem has been obtained as the limit of a sequence of solution of linear problems generated by the quasilinearization technique [10]. Finally, in the last section, the conclusions are presented.

2. JUSTIFICATION FOR USING PETROV-GALERKIN METHOD

In Finite Element Method (FEM) the approximate solution can be written as a linear combination of basis functions which constitute a basis for the approximation space under consideration. FEM involves variational methods like Rayleigh Ritz method, Galerkin method, Least Squares method, Petrov-Galerkin method and Collocation method etc. In Petrov-Galerkin method, the residual of approximation is made orthogonal to the weight functions. When we use Petrov-Galerkin method, a weak form of approximation solution for a given differential equation exists and is unique under appropriate conditions [5, 7] irrespective of properties of a given differential operator. Further, a weak solution also tends to a classical solution of given differential equation, provided sufficient attention is given to the boundary conditions [1]. That means the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are prescribed and also the number of weight functions should match with the number of basis functions. Hence in this paper we employed the use of Petrov-Galerkin method with quintic B-splines as basis functions and sextic B-splines as weight functions to approximate the solution of a seventh order boundary value problem.

3. DESCRIPTION OF THE METHOD

Definition of quintic B-splines and sextic B-splines:

The quintic B-splines and sextic B-splines are defined in [4,2,9]. The existence of quintic spline interpolate s(t) to a function in a closed interval [c, d] for spaced knots (need not be evenly spaced) of a partition $c = t_0 < t_1 < \ldots < t_{n-1} < t_n = d$ is established by constructing it. The construction of $s(t)$ is done with the help of the quintic B-splines. Introduce ten additional knots t_{-5}, t_{-4}, t_{-3}, t_{-2}, t_{-1}, t_{n+1}, t_{n+2}, t_{n+3}, t_{n+4} and t_{n+5} in such a way that

$t_{-5}<t_{-4}<t_{-3}<t_{-2}<t_{-1}<t_0$ and $t_n<t_{n+1}<t_{n+2}<t_{n+3}<t_{n+4}<t_{n+5}$.

Now the quintic B-splines $B_i(t)'s$ are defined by

$$B_i(t)=\begin{cases}\sum_{r=i-3}^{i+3}\dfrac{(t_r-t)_+^5}{\pi'(t_r)}, & t\in[t_{i-3},\, t_{i+3}]\\ 0, & otherwise\end{cases}$$

where $$(t_r-t)_+^5=\begin{cases}(t_r-t)^5, & if\ t_r\ge t\\ 0, & if\ t_r\le t\end{cases}$$

and $$\pi(t)=\prod_{r=i-3}^{i+3}(t-t_r)$$

where {$B_{-2}(t)$, $B_{-1}(t)$, $B_0(t)$, $B_1(t)$,…,$B_{n-1}(t)$, $B_n(t)$, $B_{n+1}(t)$, $B_{n+2}(t)$} forms a basis for the space $S_5(\pi)$ of quintic polynomial splines. Schoenberg [4] has proved that quintic B-splines are the unique nonzero splines of smallest compact support with the knots at

$t_{-5}<t_{-4}<t_{-3}<t_{-2}<t_{-1}<t_0<t_1<\ldots<t_{n-1}<t_n<t_{n+1}<t_{n+2}<t_{n+3}<t_{n+4}<t_{n+5}$.

In a similar analogue sextic B-splines $S_i(t)'s$ are defined by

$$S_i(t)=\begin{cases}\sum_{r=i-3}^{i+4}\dfrac{(t_r-t)_+^6}{\pi'(t_r)}, & t\in[t_{i-3},\,t_{i+4}]\\ 0, & otherwise\end{cases}$$

where $$(t_r-t)_+^6=\begin{cases}(t_r-t)^6, & if\ t_r\geq t\\ 0, & if\ t_r\leq t\end{cases}$$

and $$\pi(t)=\prod_{r=i-3}^{i+4}(t-t_r)$$

where $\{S_{-3}(t), S_{-2}(t), S_{-1}(t), S_0(t), S_1(t),\ldots, S_{n-1}(t), S_n(t), S_{n+1}(t), S_{n+2}(t)\}$ forms a basis for the space $S_6(\pi)$ of sextic polynomial splines with the introduction of two more additional knots t_{-6} and t_{n+6} to the already existing knots t_{-5} to t_{n+5}. Schoenberg [4] has proved that sextic B-splines are the unique nonzero splines of smallest compact support with the knots at

$t_{-6}<t_{-5}<t_{-4}<t_{-3}<t_{-2}<t_{-1}<t_0<t_1<\ldots<t_{n-1}<t_n<t_{n+1}<t_{n+2}<t_{n+3}<t_{n+4}<t_{n+5}<t_{n+6}$.

To solve the boundary value problem (1) subject to boundary conditions (2) by the Petrov-Galerkin method with quintic B-splines as basis functions and sextic B-splines as weight functions, we define the approximation for $u(t)$ as

$$u(t)=\sum_{j=-2}^{n+2}\alpha_j B_j(t) \tag{3}$$

where α_j*'s* are the nodal parameters to be determined and $B_j(t)$*'s* are quintic B-spline basis functions. In Petrov-Galerkin method, the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are specified. In the set of quintic B-splines $\{B_{-2}(t), B_{-1}(t), B_0(t), B_1(t),\ldots,B_{n-1}(t), B_n(t), B_{n+1}(t), B_{n+2}(t)\}$, the basis functions $B_{-2}(t), B_{-1}(t), B_0(t), B_1(t), B_2(t), B_{n-2}(t), B_{n-1}(t), B_n(t), B_{n+1}(t)$ and $B_{n+2}(t)$ do not vanish at one of the boundary points. So, there is a necessity of redefining the basis functions into a new set of basis functions which vanish on the boundary where the Dirichlet type of boundary conditions are specified. When the chosen approximation satisfies the prescribed boundary conditions or most of the boundary conditions, it gives better approximation results. In view of this, the basis functions are redefined into a new set of basis functions which vanish on the boundary where the Dirichlet, the Neumann type of boundary conditions and the second order derivative boundary condition at the left boundary are prescribed. The procedure for redefining of the basis functions is as follows.

Using the definition of quintic B-splines, the Dirichlet, the Neumann boundary conditions and the second order derivative boundary condition at the left boundary of (2), we get the approximate solution at the boundary points as

$$A_0=u(c)=u(t_0)=\sum_{j=-2}^{2}\alpha_j B_j(t_0) \tag{4}$$

$$C_0 = u(d) = u(t_n) = \sum_{j=n-2}^{n+2} \alpha_j B_j(t_n) \tag{5}$$

$$A_1 = u'(c) = u'(t_0) = \sum_{j=-2}^{2} \alpha_j B_j'(t_0) \tag{6}$$

$$C_1 = u'(d) = u'(t_n) = \sum_{j=n-2}^{n+2} \alpha_j B_j'(t_n) \tag{7}$$

$$A_2 = u''(c) = u''(t_0) = \sum_{j=-2}^{2} \alpha_j B_j''(t_0) \tag{8}$$

Eliminating α_{-2}, α_{-1}, α_0, α_{n+1} and α_{n+2} from the equations (3) to (8), we get

$$u(t) = w(t) + \sum_{j=1}^{n} \alpha_j R_j(t) \tag{9}$$

where

$$w(t) = w_2(t) + \frac{A_2 - w_2''(t_0)}{Q_0''(t_0)} Q_0(t) \tag{10}$$

$$w_2(t) = w_1(t) + \frac{A_1 - w_1'(t_0)}{P_{-1}'(t_0)} P_{-1}(t) + \frac{C_1 - w_1'(t_n)}{P_{n+1}'(t_n)} P_{n+1}(t) \tag{11}$$

$$w_1(t) = \frac{A_0}{B_{-2}(t_0)} B_{-2}(t) + \frac{C_0}{B_{n+2}(t_n)} B_{n+2}(t) \tag{12}$$

$$R_j(t) = \begin{cases} Q_j(t) - \dfrac{Q_j''(t_0)}{Q_0''(t_0)} Q_0(t), & j = 1, 2 \\ Q_j(t), & j = 3, 4, \ldots, n \end{cases} \tag{13}$$

$$Q_j(t) = \begin{cases} P_j(t) - \dfrac{P_j'(t_0)}{P_{-1}'(t_0)} P_{-1}(t), & j = 0, 1, 2 \\ P_j(t), & j = 3, 4, \ldots, n-3 \\ P_j(t) - \dfrac{P_j'(t_n)}{P_{n+1}'(t_n)} P_{n+1}(t), & j = n-2, n-1, n \end{cases} \tag{14}$$

$$P_j(t) = \begin{cases} B_j(t) - \dfrac{B_j(t_0)}{B_{-2}(t_0)} B_{-2}(t), & j = -1, 0, 1, 2 \\ B_j(t), & j = 3, 4, \ldots, n-3 \\ B_j(t) - \dfrac{B_j(t_n)}{B_{n+2}(t_n)} B_{n+2}(t), & j = n-2, n-1, n, n+1 \end{cases} \tag{15}$$

The new set of basis functions in the approximation $u(t)$ is $\{R_j(t), j=1,2,\ldots,n\}$. Here $w(t)$ takes care of given set of the Dirichlet, the Neumann type boundary conditions and the second order derivative boundary condition at the left boundary. $R_j(t)$'s and its first order derivatives, second order derivative at left vanish on the boundary. In Petrov-Galerkin method, the number of basis functions in the approximation should match with the number of weight functions. Here the number of basis functions in the approximation for $u(t)$ defined in (9) is n, where as the number of weight functions is $n+6$. So, there is a need to redefine the weight functions into a new set of weight functions which in number match with the number of basis functions. The procedure for redefining the weight functions is as follows.

Let us write the approximation for $v(t)$ as

$$v(t) = \sum_{j=-3}^{n+2} \beta_j S_j(t) \tag{16}$$

where $S_j(t)$'s are sextic B-splines and here we assume that above approximation $v(t)$ satisfies corresponding homogeneous boundary conditions of the Dirichlet, Neumann and second order derivative boundary conditions given in (2). That means $v(t)$ defined in (16) satisfies the conditions

$$v(c)=0, v(d)=0, v'(c)=0, v'(d)=0, v''(c)=0, v''(d)=0 \tag{17}$$

Applying the boundary conditions (17) to (16), we get the approximate solution at the boundary points as

$$v(c) = v(t_0) = \sum_{j=-3}^{2} \beta_j S_j(t_0) = 0 \tag{18}$$

$$v(d) = v(t_n) = \sum_{j=n-3}^{n+2} \beta_j S_j(t_n) = 0 \tag{19}$$

$$v'(c) = v'(t_0) = \sum_{j=-3}^{2} \beta_j S_j'(t_0) = 0 \tag{20}$$

$$v'(d) = v'(t_n) = \sum_{j=n-3}^{n+2} \beta_j S_j'(t_n) = 0 \tag{21}$$

$$v''(c) = v''(t_0) = \sum_{j=-3}^{2} \beta_j S_j''(t_0) = 0 \tag{22}$$

$$v''(d) = v''(t_n) = \sum_{j=n-3}^{n+2} \beta_j S_j''(t_n) = 0 \quad (23)$$

Eliminating β_{-3}, β_{-2}, β_{-1}, β_n, β_{n+1} and β_{n+2} from the equations (16) and (18) to (23), we get the approximation for *v(t)* as

$$v(t) = \sum_{j=0}^{n-1} \beta_j V_j(t) \quad (24)$$

where

$$V_j(t) = \begin{cases} U_j(t) - \dfrac{U_j''(t_0)}{U_{-1}''(t_0)} U_{-1}(t), & j = 0,1,2 \\ U_j(t), & j = 3,4,5,\ldots,n-4 \\ U_j(t) - \dfrac{U_j''(t_n)}{U_n''(t_n)} U_n(t), & j = n-3, n-2, n-1 \end{cases} \quad (25)$$

$$U_j(t) = \begin{cases} T_j(t) - \dfrac{T_j'(t_0)}{T_{-2}'(t_0)} T_{-2}(t), & j = -1,0,1,2 \\ T_j(t), & j = 3,4,5,\ldots,n-4 \\ T_j(t) - \dfrac{T_j'(t_n)}{T_{n+1}'(t_n)} T_{n+1}(t), & j = n-3, n-2, n-1, n \end{cases} \quad (26)$$

$$T_j(t) = \begin{cases} S_j(t) - \dfrac{S_j(t_0)}{S_{-3}(t_0)} S_{-3}(t), & j = -2,-1,0,1,2 \\ S_j(t), & j = 3,4,5\ldots,n-4 \\ S_j(t) - \dfrac{S_j(t_n)}{S_{n+2}(t_n)} S_{n+2}(t), & j = n-3, n-2, n-1, n, n+1 \end{cases} \quad (27)$$

Now the new set of weight functions for the approximation *v(t)* is $\{V_j(t), j=0,1,\ldots,n-1\}$. Here $V_j(t)$*'s* and its first and second order derivatives vanish on the boundary.

Applying the Petrov-Galerkin method to (1) with the new set of basis functions $\{R_j(t), j=1,\ldots,n\}$ and with the new set of weight functions $\{V_j(t), j=0,1,\ldots,n-1\}$, we get

$$\int_{t_0}^{t_n} [p_0(t)u^{(7)}(t) + p_1(t)u^{(6)}(t) + p_2(t)u^{(5)}(t) + p_3(t)u^{(4)}(t) + \\ p_4(t)u'''(t) + p_5(t)u''(t) + p_6(t)u'(t) + p_7(t)u(t)]V_i(t)\,dt = \int_{t_0}^{t_n} b(t)V_i(t)dt \quad \text{for } i = 0, 1, \ldots, n\text{-}1. \quad (28)$$

Integrating by parts the first three terms on the left hand side of (28) and after applying the boundary conditions prescribed in (2), we get

$$\int_{t_0}^{t_n} p_0(t)u^{(7)}(t)V_i(t)dt = -\frac{d^3}{dt^3}\big[p_0(t)V_i(t)\big]_{t_n} u'''(t_n) + \frac{d^3}{dt^3}\big[p_0(t)V_i(t)\big]_{t_0} A_3$$
$$+\frac{d^4}{dt^4}\big[p_0(t)V_i(t)\big]_{t_n} C_2 - \frac{d^4}{dt^4}\big[p_0(t)V_i(t)\big]_{t_0} A_2 - \int_{t_0}^{t_n} \frac{d^5}{dt^5}\big[p_0(t)V_i(x)\big]u''(t)\,dt \tag{29}$$

$$\int_{t_0}^{t_n} p_1(t)u^{(6)}(t)V_i(t)dt = -\frac{d^3}{dt^3}\big[p_1(t)V_i(t)\big]_{t_n} C_2 + \frac{d^3}{dt^3}\big[p_1(t)V_i(t)\big]_{t_0} A_2$$
$$+\int_{t_0}^{t_n} \frac{d^4}{dt^4}\big[p_1(t)V_i(t)\big]u''(t)\,dt \tag{30}$$

$$\int_{t_0}^{t_n} p_2(t)u^{(5)}(t)V_i(t)dt = -\int_{t_0}^{t_n} \frac{d^3}{dt^3}\big[p_2(t)V_i(t)\big]u''(t)\,dt \tag{31}$$

Substituting (29), (30) and (31) in (28) and using the approximation for $u(t)$ given in (9), and after rearranging the terms for resulting equations, we get a system of equations in the matrix form as

$$\mathbf{A}\alpha = \mathbf{B} \tag{32}$$

where $\mathbf{A} = [a_{ij}]$;

$$a_{ij} = \int_{t_0}^{t_n} \{p_3(t)V_i(t)R_j^{(4)}(t) + p_4(t)V_i(t)R_j'''(t) + [-\frac{d^5}{dt^5}[p_0(t)V_i(t)]$$
$$+\frac{d^4}{dt^4}[p_1(t)V_i(t)] - \frac{d^3}{dt^3}[p_2(t)V_i(t)] + p_5(t)V_i(t)]R_j''(t)$$
$$+ p_6(t)V_i(t)R_j'(t) + p_7(t)V_i(t)R_j(t)\}dt$$
$$-\frac{d^3}{dt^3}[p_0(t)V_i(t)]_{t_n} R_j'''(t_n)$$

for i= 0,1,..., n-1;j= 1,2,..., n. (33)

$\mathbf{B}=[b_i];$

$$b_i=\int_{t_0}^{t_n}\{b(t)V_i(t)-p_3(t)V_i(t)w^{(4)}(t)-p_4(t)V_i(t)w'''(t)-[-\frac{d^5}{dt^5}[p_0(t)V_i(t)]$$
$$+\frac{d^4}{dt^4}[p_1(t)V_i(t)]-\frac{d^3}{dt^3}[p_2(t)V_i(t)]+p_5(t)V_i(t)]w''(t)$$
$$-p_6(t)V_i(t)w'(t)-p_7(t)V_i(t)w(t)\}dt+\frac{d^3}{dt^3}[p_0(t)V_i(t)]_{t_n}w'''(t_n)$$
$$-\frac{d^3}{dt^3}[p_0(t)V_i(t)]_{t_0}A_3-\frac{d^4}{dt^4}[p_0(t)V_i(t)]_{t_n}C_2+\frac{d^4}{dt^4}[p_0(t)V_i(t)]_{t_0}A_2$$
$$+\frac{d^3}{dt^3}[p_1(t)V_i(t)]_{t_n}C_2-\frac{d^3}{dt^3}[p_1(t)V_i(t)]_{t_0}A_2$$

for i=0,1,..., n-1. (34)

and $\alpha=[\alpha_1\,\alpha_2\ldots\alpha_n]^T$.

4. PROCEDURE TO FIND THE SOLUTION FOR NODAL PARAMETERS

A typical integral element in the matrix $\mathbf{A}$ is

$$\sum_{m=0}^{n-1} I_m$$

where $I_m=\int_{t_m}^{t_{m+1}} v_i(t)r_j(t)Z(t)dt$, $r_j(t)$ are the quintic B-spline basis functions or their derivatives, $v_i(t)$ are the sextic B-spline weight functions or their derivatives. It may be noted that $I_m = 0$ if $(t_{i-3},t_{i+4})\cap(t_{j-3},t_{j+3})\cap(t_m,t_{m+1})=\varnothing$. To evaluate each I_m, we employed 7-point Gauss-Legendre quadrature formula. Thus the stiffness matrix $\mathbf{A}$ is a twelve diagonal band matrix. The nodal parameter vector α has been obtained from the system $\mathbf{A}\alpha=\mathbf{B}$ using the band matrix solution package. We have used the FORTRAN-90 program to solve the boundary value problems (1) - (2) by the proposed method.

5. NUMERICAL RESULTS

To demonstrate the applicability of the proposed method for solving the seventh order boundary value problems of the type (1) and (2), we considered three linear and three nonlinear boundary value problems. The obtained numerical results for each problem are presented in tabular forms and compared with the exact solutions available in the literature.

Example 1: Consider the linear boundary value problem

$$u^{(7)}+u=-(35+12t+2t^2)e^t,\quad 0<t<1 \tag{35}$$

subject to

$$u(0)=0,\ \ u(1)=0, u'(0)=1, u'(1)=-e, u''(0)=0, u''(1)=-4e, u'''(0)=-3.$$

The exact solution for the above problem is $u=t(1-t)e^t$.

The proposed method is tested on this problem where the domain [0, 1] is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 1. The maximum absolute error obtained by the proposed method is 5.906817x10^{-7}.

Table 1: Numerical results for Example 1

t	Absolute error by the proposed method
0.1	2.010481E-07
0.2	1.516223E-07
0.3	1.066816E-07
0.4	2.379769E-07
0.5	4.081994E-07
0.6	5.511373E-07
0.7	5.906817E-07
0.8	4.726362E-07
0.9	2.190313E-07

Example 2: Consider the linear boundary value problem

$$u^{(7)}-tu=(t^2-2t-6)e^t,\quad 0\le t\le 1 \tag{36}$$

subject to

$$u(0)=1, u(1)=0, u'(0)=0, u'(1)=-e, u''(0)=-1, u''(1)=-2e, u'''(0)=-2.$$

The exact solution for the above problem is $u=(1-t)e^t$.

The proposed method is tested on this problem where the domain [0, 1] is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 2. The maximum absolute error obtained by the proposed method is 1.016248x10^{-6}.

Table 2: Numerical results for Example 2

t	Absolute error by the proposed method
0.1	1.016248E-06
0.2	6.373203E-07
0.3	2.263725E-08
0.4	2.497613E-08
0.5	8.782863E-08
0.6	1.664567E-07
0.7	2.191311E-07
0.8	2.015355E-07
0.9	1.031566E-07

Example 3: Consider the linear boundary value problem

$$u^{(7)} + \sin t\, u^{(4)} + \cos t\, u''' + (1-t)u = (2+\sin t+\cos t - t)e^{t}, \quad 0<t<1 \tag{37}$$

subject to

$$u(0)=1, u(1)=e, u'(0)=1, u'(1)=e, u''(0)=1, u''(1)=e, u'''(0)=1.$$

The exact solution for the above problem is $u = e^{t}$.

The proposed method is tested on this problem where the domain [0, 1] is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 3. The maximum absolute error obtained by the proposed method is 1.254922×10^{-5}.

Table 3: Numerical results for Example3

t	Absolute error by the proposed method
0.1	1.254922E-05
0.2	8.020103E-06
0.3	6.065368E-07
0.4	3.583312E-07
0.5	1.772284E-07
0.6	2.266765E-07
0.7	6.432294E-07
0.8	7.766962E-07
0.9	4.610300E-07

Example 4: Consider the nonlinear boundary value problem

$$u^{(7)} - uu' = e^{-2t}(2+e^{t}(t-8)-3t+t^{2}), \quad 0 \le t \le 1 \tag{38}$$

subject to

$$u(0)=1, u(1)=0, u'(0)=-2, u'(1)=-e^{-1}, u''(0)=3, u''(1)=2e^{-1}, u'''(0)=-4.$$

The exact solution for the above problem is $u = (1-t)e^{-t}$.

The nonlinear boundary value problem (38) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [10] as

$$u_{(n+1)}^{(7)} - u_{(n)}u'_{(n+1)} - u'_{(n)}u_{(n+1)} = e^{-2t}(2 + e^{t}(t-8) - 3t + t^2) - u_{(n)}u'_{(n)}, \quad n = 0,1,2,... \tag{39}$$

subject to

$$u_{(n+1)}(0) = 1,\, u_{(n+1)}(1) = 0,\, u'_{(n+1)}(0) = -2,\, u'_{(n+1)}(1) = -e^{-1},$$
$$u''_{(n+1)}(0) = 3, u''_{(n+1)}(1) = 2e^{-1},\, u'''_{(n+1)}(0) = -4.$$

Here $u_{(n+1)}$ is the $(n+1)^{th}$ approximation for $u(t)$. The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (39). The obtained numerical results for this problem are presented in Table 4. The maximum absolute error obtained by the proposed method is 6.887752x10^{-6}.

Table 4: Numerical results for Example 4

t	Absolute error by the proposed method
0.1	6.887752E-06
0.2	4.819232E-06
0.3	1.725417E-06
0.4	2.661401E-06
0.5	3.213295E-06
0.6	2.900484E-06
0.7	1.885416E-06
0.8	7.222064E-07
0.9	4.172772E-08

Example 5: Consider the nonlinear boundary value problem

$$u^{(7)} + u^{(4)} - e^{u}u = e^{t}(12\sin t + 40\cos t + 8t\cos t - 4t\sin t - (1-t)\sin t\; e^{e^{t}(1-t)\sin t}), \quad 0 < t < 1 \tag{40}$$

subject to

$$u(0) = 0,\, u(1) = 0,\, u'(0) = 1,\, u'(1) = -esin1, u''(0) = 0,\, u''(1) = -2ecos1 - 2esin1,$$
$$u'''(0) = -4.$$

The exact solution for the above problem is $y = e^{t}(1-t)\sin t$.

The nonlinear boundary value problem (40) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [10] as

$$u_{(n+1)}^{(7)} + u_{(n+1)}^{(4)} - e^{u_{(n)}}(1+u_{(n)})u_{(n+1)} = e^t(12\sin t + 40\cos t + 8t\cos t - 4t\sin t$$
$$-(1-t)\sin t\; e^{e^t(1-t)\sin t}) - e^{u_{(n)}}u_{(n)}^2 \qquad n = 0,1,2,\ldots \tag{41}$$

subject to

$$u_{(n+1)}(0) = 0,\ u_{(n+1)}(1) = 0,\ u'_{(n+1)}(0) = 1,\ u'_{(n+1)}(1) = -esin1,$$
$$u''_{(n+1)}(0) = 0, u''_{(n+1)}(1) = -2ecos1 - 2esin1, u'''_{(n+1)}(0) = -4.$$

Here $u_{(n+1)}$ is the $(n+1)^{th}$ approximation for $u(t)$. The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (41). The obtained numerical results for this problem are presented in Table 5. The maximum absolute error obtained by the proposed method is 4.067081x10^{-6}.

Table 5: Numerical results for Example 5

t	Absolute error by the proposed method
0.1	1.993766E-06
0.2	1.428117E-06
0.3	7.466525E-07
0.4	1.630706E-06
0.5	2.801159E-06
0.6	3.788677E-06
0.7	4.067081E-06
0.8	3.258684E-06
0.9	1.511690E-06

Example 6: Consider the nonlinear boundary value problem

$$u^{(7)} + \sin u\; u^{(4)} + e^u u'' = e^t(1 + sin(e^t) + e^{e^t}), \quad 0 < t < 1 \tag{42}$$

subject to

$$u(0) = 1, u'(0) = e, u(1) = 1, u'(1) = e, u''(0) = 1, u''(1) = e, u'''(0) = 1.$$

The exact solution for the above problem is $u = e^t$.

The nonlinear boundary value problem (42) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [10] as

$$\begin{aligned} &u_{(n+1)}^{(7)} + sin(u_{(n)})u_{(n+1)}^{(4)} + e^{u_{(n)}}u''_{(n+1)} + (\cos(u_{(n)})u_{(n)}^{(4)} + e^{u_{(n)}}u''_{(n)})u_{(n+1)} \\ &= (\cos(u_{(n)})u_{(n)}^{(4)} + e^{u_{(n)}}u''_{(n)})u_{(n)} + e^t(1 + sin(e^t) + e^{e^t}), \quad n = 0,1,2,\ldots \end{aligned} \tag{43}$$

subject to

$$u_{(n+1)}(0) = 1, u_{(n+1)}(1) = e, u'_{(n+1)}(0) = 1, u'_{(n+1)}(1) = e, u''_{(n+1)}(0) = 1, u''_{(n+1)}(1) = e, u'''_{(n+1)}(0) = 1.$$

Here $u_{(n+1)}$ is the $(n+1)^{th}$ approximation for u(t). The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (43). The obtained numerical results for this problem are presented in Table 6. The maximum absolute error obtained by the proposed method is 1.254944×10^{-5}.

Table 6: Numerical results for Example 6

t	Absolute error by the Proposed method
0.1	1.254944E-05
0.2	8.016145E-06
0.3	5.829692E-07
0.4	3.018141E-07
0.5	9.353160E-08
0.6	3.169775E-07
0.7	7.179021E-07
0.8	8.230447E-07
0.9	4.803181E-07

6. CONCLUSIONS

In this paper, we have employed a Petrov-Galerkin method with quintic B-splines as basis functions and sextic B-splines as weight functions to solve a general seventh order boundary value problems with special case of boundary conditions. The quintic B-spline basis set has been redefined into a new set of basis functions which vanish on the boundary where the Dirichlet, the Neumann boundary conditions and the second order derivative boundary condition at the left boundary are prescribed. The sextic B-splines are redefined into a new set of weight functions which in number match the number of redefined set of basis functions. The solution to a nonlinear problem has been obtained as the limit of a sequence of solution of linear problems generated by the quasilinearization technique [10]. The proposed method has been tested on three linear and three nonlinear seventh order boundary value problems. The numerical results obtained by the proposed method are in good agreement with the exact solutions available in the literature. The strength of the proposed method lies in its easy applicability, accurate and efficient to solve seventh order boundary value problems.

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