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A PROPOSED ONE-STEP EIGHTH-ORDER NUMERICAL METHOD USING PADE APPROXIMANTS FOR THE SOLUTIONS OF STIFF DIFFERENTIAL EQUATIONS

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Abstract

This paper proposes a new one-step eighth-order method for solving stiff differential equations by using pade rational functions. The results show that the method is consistent and convergent. L-stability of the method is identified. Numerical results and comparative analysis with exact solution show that the method is efficient and accurate.

Keywords:

Stiff ODE, Convergence, L-stability, LTE.

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I. Introduction

We consider the initial value problem given by

$$y' = f(x,y),\ y(x_0) = y_0,\ a \le x \le b \qquad \text{... (1)}$$

for one-step numerical method with the order K=8. This class of problem has a lot of applications in many areas such as chemistry, electrical engineering and biological systems etc.

Definition 1.1:

A system of ordinary differential equations of the form

$y' = f(x, y_1, y_2, y_3, \ldots, y_m)$, $y(x_0) = y_0$
... (2)

is said to be stiff if the eigen value λ_i of the Jucobian matrix $\left[\frac{\partial f}{\partial y}\right]$ at every integration point x have negative real parts and differ greatly in magnitude.

Also, the eigen values λ_t satisfy the following conditions [6].

(i) $Re(\lambda_t) < 0$, t = 1,2, ... , m and

(ii) $\frac{\max_t |\lambda_t|}{\min_t |\lambda_t|} = S > 1$; S is the stiffness ratio.

Most familiar numerical methods such as Euler method, Runge-Kutta methods etc., cannot solve stiff problems as they possess least stability properties. Butcher [3], Fatunla [5] and many

others have developed L-stable methods for solving stiff initial value problems in ordinary differential equations for this reason.
Liniger and Wolloigbby (1970) as cited in [1] introduced the concept of exponential fitting and suggested three new A-stable methods with K = 1. Abhulimen and Ukpebor [2] developed exponentially fitted multiderivative methods. Abhulimen and Otunta [1] developed sixth order multiderivative multistep methods with step number K = 2. A one-step, sixth order method for solving stiff problems was developed in Umar Ahmad Egbako and Kayode R.Adeboye [7].

The aim of this paper is to derive an efficient eighth order, one step, L-stable method based on pade rational approximations.

2. Derivation of the method

We consider pade rational approximant of the form:

$$F_N^M(x) = \frac{\sum_{i=0}^{M} a_i x^i}{1+\sum_{j=1}^{N} b_j x^j}, M \geq 0, N \geq 0, i = 0,1(M), j = 1,2(N) \qquad \text{... (3)}$$

where a_i, b_j are real coefficients as cited in [7] .

A finite difference numerical integrator of maximal order K = M + N approximately to the initial value problem is defined as follows:

$y' = f(x, y), y(x_0) = y_0$ by

$$y_{n+1} = \frac{\sum_{i=0}^{M} a_i x^i}{1+\sum_{j=1}^{N} b_j x^j} \qquad \text{... (4)}$$

with M ≥ 0, N ≥ 0, i = 0, 1(M) and j = 1,2(N), The parameters $\{a_i\}$ and $\{b_j\}$ are constant coefficients to be determined.

To develop a one-step, eight-order method we set M = 4, N = 4 in equation (4):

$$y_{n+1} = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4}{1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4}. \qquad \text{... (5)}$$

By the Taylor series of y_{n+1} and ignoring the terms of order higher than 6 in equation (5), we obtain,

$$\frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4}{1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4} = y_n + h y_n' + \frac{h^2 y_n''}{2!} + \frac{h^3 y_n'''}{3!} + \frac{h^4 y_n^{iv}}{4!} + \frac{h^5 y_n^{v}}{5!} + \frac{h^6 y_n^{vi}}{6!} + O(h^7)$$

$$\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right) = \left(1 + b_1 x + b_2 x^2 + b_3 x^3\right)\left(y_n + h y_n^1 + \frac{h^2 y_n''}{2!} + \frac{h^3 y_n'''}{3!} + \frac{h^4 y_n^{iv}}{4!} + \frac{h^5 y_n^{v}}{5!} + \frac{h^6 y_n^{vi}}{6!}\right) + O(h^7)$$

Taking x = h and equating the coefficients as far as h^8 we have,

$$a_0 = y_n \qquad \text{... (6)}$$

$$a_1 = b_1 y_n + h y_n^1 \qquad \text{... (7)}$$

$$a_2 = b_2 y_n + h b_1 y_n' + \frac{h^2 y_n''}{2} \qquad \text{... (8)}$$

$$a_3 = b_3 y_n + h b_2 y_n' + \frac{b_1 h^2 y_n''}{2} + \frac{h^3 y_n'''}{6} \qquad \text{... (9)}$$

$$a_4 = b_4 y_n + b_3 h y_n' + \frac{b^2 h^2 y_n''}{2} + \frac{b_1 h^3 y_n'''}{6} + \frac{h^4 y_n^{(iv)}}{24} \qquad \text{... (10)}$$

$$b_4 h y_n' + \frac{b_3 h^2 y_n''}{2} + \frac{b_2 h^3 y_n'''}{6} + \frac{b_1 h^4 y_n^{(iv)}}{24} + \frac{h^5 y_n^{(v)}}{120} = 0 \qquad \text{... (11)}$$

$$b_4 h^2 y_n'' + \frac{b_3 h^3 y_n'''}{6} + \frac{b_2 h^4 y_n^{(iv)}}{24} + \frac{b_1 h^5 y_n^{(v)}}{120} + \frac{h^6 y_n^{(vi)}}{720} = 0 \qquad \text{... (12)}$$

$$\frac{b_4 h^3 y_n'''}{6} + \frac{b_3 h^4 y_n^{(iv)}}{24} + \frac{b_2 h^5 y_n^{(v)}}{120} + \frac{b_1 h^6 y_n^{(vi)}}{720} + \frac{h^7 y_n^{(vii)}}{5040} = 0 \qquad \text{... (13)}$$

$$\frac{b_4 h^4 y_n^{(iv)}}{24} + \frac{b_3 h^5 y_n^{(v)}}{120} + \frac{b_2 h^6 y_n^{(vi)}}{720} + \frac{b_1 h^7 y_n^{(vii)}}{5040} + \frac{h^8 y_n^{(viii)}}{40320} = 0 \quad \text{... (14)}$$

Solving for b_1, b_2, b_3 and b_4 in (11), (12), (13), (14) we have.

$$\begin{bmatrix} 120hy_n' & 60h^2 y_n'' & 20h^3 y_n''' & 5h^4 y_n^{(iv)} \\ 360h^2 y_n'' & 120h^3 y_n''' & 30h^4 y_n^{(iv)} & 6h^5 y_n^{(v)} \\ 840h^3 y_n''' & 210h^4 y_n^{(iv)} & 42h^5 y_n^{(v)} & 7h^6 y_n^{(vi)} \\ 1680h^4 y_n^{(iv)} & 336h^5 y_n^{(v)} & 56h^6 y_n^{(vi)} & 8h^7 y_n^{(viii)} \end{bmatrix} \begin{bmatrix} b_4 \\ b_3 \\ b_3 \\ b_1 \end{bmatrix} = \begin{bmatrix} -h^5 y_n^{(v)} \\ -h^6 y_n^{(vi)} \\ -h^7 y_n^{(vii)} \\ -h^8 y_n^{(viii)} \end{bmatrix}$$

$$b_4 = \frac{12h^{20} S}{20160 h^{16} U}$$

$$b_4 = \frac{h^4 S}{1680 U} \quad \text{... (15)}$$

where $U = U_1 + U_2 + U_3 + U_4$,
$S = S_1 + S_2 + S_3 + S_4$

$$U_1 = 4y_n'[10y_n'''(6y_n^{(v)} y_n^{(vii)} - 7y_n^{(vi)^2}) - 15y_n^{(iv)}(5y_n^{(iv)} y_n^{(vii)} - 7y_n^{(v)} y_n^{(vi)} + 21y_n^{(v)}(5y_n^{(iv)} y_n^{(vi)} - 6y_n^{(v)^2})]$$

$$U_2 = -30y_n''[2y_n''(6y_n^{(v)} y_n^{(vii)} - 7y_n^{(vi)^2}) - 5y_n^{(iv)}(4y_n^{(iii)} y_n^{(viii)} - 7y_n^{(iv)} y_n^{(vi)}) + 14y_n^{(v)}(2y_n''' y_n^{(vi)} - 3y_n^{(iv)} y_n^{(v)})]$$

$$U_3 = 20y_n'''[6y_n''(5y_n^{(iv)} y_n^{(vii)} - 7y_n^{(v)} y_n^{(vi)}) - 10y_n'''(4y_n''' y_n^{(vii)} - 7y_n^{(iv)} y_n^{(vi)}) + 21y_n^{(v)}(4y_n''' y_n^{(v)} - 5y_n^{(iv)^2})]$$

$$U_4 = -35y_n^{(iv)}[6y_n''(5y_n^{(iv)} y_n^{(vi)} - 6y_n^{(v)^2}) - 20y_n'''(2y_n''' y_n^{(vi)} - 3y_n^{(iv)} y_n^{(v)}) + 15y_n^{(iv)}(4y_n''' y_n^{(v)} - 5y_n^{(iv)^2})] \quad \text{...}$$

(16)

$$S_1 = -56y_n^{(v)}[10y_n'''(6y_n^{(v)} y_n^{(vii)} - 7y_n^{(vi)^2}) - 15y_n^{(iv)}(5y_n^{(iv)} y_n^{(vii)} - 7y_n^{(v)} y_n^{(vi)}) - 21y_n^{(v)}(5y_n^{(iv)} y_n^{(vi)} - 6y_n^{(v)^2})]$$

$$S_2 = -10y_n''[-28y_n^{(vi)}(6y_n^{(v)} y_n^{(vii)} - 7y_n^{(vi)^2}) - 15y_n^{(iv)}(-8y_n^{(vii)^2} + 7y_n^{(viii)} y_n^{(vi)}) - 42y_n^{(v)}(-4y_n^{(vii)} y_n^{(vi)} + 3y_n^{(viii)} y_n^{(v)})]$$

$$S_3 = 20y_n'''[-28y_n^{(vi)}(5y_n^{(iv)} y_n^{(vii)} - 7y_n^{(v)} y_n^{(vi)}) - 10y_n'''(-8y_n^{(vii)^2} + 7y_n^{(viii)} y_n^{(vi)}) + 21y_n^{(v)}(-8y_n^{(vii)} y_n^{(v)} + 5y_n^{(viii)} y_n^{(iv)})]$$

$$S_4 = y_n^{(iv)}[-28y_n^{(vi)}(5y_n^{(iv)} y_n^{(vi)} - 6y_n^{(v)^2}) - 20y_n'''(-4y_n^{(vii)} y_n^{(vi)} + 3y_n^{(viii)} y_n^{(v)}) + 15y_n^{(iv)}(-8y_n^{(iv)} y_n^{(v)} + 5y_n^{(viii)} y_n^{(iv)})] \quad \text{...}$$

(17)

$$b_3 = \frac{240h^{19} R}{20160 h^{16} U}$$

$$b_3 = \frac{h^3 R}{84U} \quad \text{... (18)}$$

where $R = R_1 + R_2 + R_3 + R_4$

$$R_1 = y_n'[-28y_n^{(vi)}(6y_n^{(v)} y_n^{(vii)} - 7y_n^{(vi)^2}) - 15y_n^{(iv)}(-8y_n^{(vii)^2} + 7y_n^{(viii)} y_n^{(vi)}) + 42y_n^{(v)}(-4y_n^{(vii)} y_n^{(vi)} + 3y_n^{(viii)} y_n^{(v)})]$$

$$R_2 = 42y_n^{(v)}[2y_n''(6y_n^{(v)} y_n^{(vii)} - 7y_n^{(vi)^2}) - 5y_n^{(iv)}(4y_n''' y_n^{(vii)} - 7y_n^{(iv)} y_n^{(vi)}) + 14y_n^{(v)}(2y_n''' y_n^{(vi)} - 3y_n^{(iv)} y_n^{(v)})]$$

$$R_3 = 10y_n'''[3y_n''(-8y_n^{(vii)^2} + 7y_n^{(viii)} y_n^{(vi)}) + 14y_n^{(vi)}(4y_n''' y_n^{(vii)} - 7y_n^{(iv)} y_n^{(vi)}) + 42y_n^{(v)}(-y_n''' y_n^{(viii)} + 2y_n^{(iv)} y_n^{(vii)})]$$

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$$R_4 = -35y_n^{(iv)}[3y_n''(-4y_n^{(vii)}y_n^{(vi)} + 3y_n^{(viii)}y_n^{(v)}) + 14y_n^{(vi)}(2y_n'''y_n^{(vi)} - 3y_n^{(iv)}y_n^{(v)}) + 15y_n^{(iv)}(-y_n'''y_n^{(viii)} + 2y_n^{(iv)}y_n^{(vii)})] \quad \ldots (19)$$

$$b_2 = \frac{720h^{18}T}{20160h^{16}U}$$

$$b_2 = \frac{h^2T}{28U} \quad \ldots (20)$$

where T = $T_1 + T_2 + T_3 + T_4$

$$T_1 = 2y_n'[10y_n'''(-8y_n^{(vii)^2} + 7y_n^{(viii)}y_n^{(vi)}) + 28y_n^{(vi)}(5y_n^{(iv)}y_n^{(vii)} - 7y_n^{(v)}y_n^{(vi)}) + 21y_n^{(v)}(-5y_n^{(iv)}y_n^{(viii)} + 8y_n^{(v)}y_n^{(vi)})]$$

$$T_2 = -10y_n''[3y_n''(-8y_n^{(vii)^2} + 7y_n^{(viii)}y_n^{(vi)}) + 14y_n^{(vi)}(4y_n'''y_n^{(vii)} - 7y_n^{(iv)}y_n^{(vi)}) + 42y_n^{(v)}(-y_n'''y_n^{(viii)} + 2y_n^{(iv)}y_n^{(vii)})]$$

$$T_3 = -28y_n^{(v)}[6y_n''(5y_n^{(iv)}y_n^{(vii)} - 7y_n^{(v)}y_n^{(vi)}) - 10y_n'''(4y_n'''y_n^{(vii)} - 7y_n^{(iv)}y_n^{(vi)}) + 21y_n^{(v)}(4y_n'''y_n^{(v)} - 5y_n^{(iv)^2})]$$

$$T_4 = -35y_n^{(iv)}[3y_n''(5y_n^{(iv)}y_n^{(viii)} + 8y_n^{(v)}y_n^{(vii)}) - 20y_n'''(-y_n'''y_n^{(viii)} + 2y_n^{(vi)}y_n^{(vii)}) - 14y_n^{(vi)}(4y_n'''y_n^{(v)} - 5y_n^{(iv)^2})] \quad \ldots (21)$$

$$b_1 = \frac{10080h^{17}V}{20160h^{16}U}$$

$$b_1 = \frac{hV}{12U} \quad \ldots (22)$$

where V = $V_1 + V_2 + V_3 + V_4$

$$V_1 = y_n'[20y_n'''(-3y_n^{(v)}y_n^{(viii)} + 4y_n^{(vi)}y_n^{(vii)}) - 15y_n^{(iv)}(-5y_n^{(iv)}y_n^{(viii)} + 8y_n^{(v)}y_n^{(vii)}) - 28y_n^{(vi)}(5y_n^{(iv)}y_n^{(vi)} - 6y_n^{(v)^2})]$$

$$V_2 = -10y_n''[3y_n''(-3y_n^{(v)}y_n^{(viii)} + 4y_n^{(vi)}y_n^{(vii)}) - 15y_n^{(iv)}(-y_n'''y_n^{(viii)} + 2y_n^{(iv)}y_n^{(vii)}) - 14y_n^{(vi)}(2y_n'''y_n^{(vi)} - 3y_n^{(iv)}y_n^{(v)})]$$

$$V_3 = -10y_n'''[3y_n''(-5y_n^{(iv)}y_n^{(viii)} + 8y_n^{(v)}y_n^{(vii)}) - 20y_n'''(-y_n'''y_n^{(viii)} + 2y_n^{(iv)}y_n^{(vii)}) - 14y_n^{(vi)}(4y_n'''y_n^{(v)} - 5y_n^{(iv)^2})]$$

$$V_4 = 14y_n^{(v)}[6y_n''(5y_n^{(iv)}y_n^{(vi)} - 6y_n^{(v)^2}) - 20y_n'''(2y_n'''y_n^{(vi)} - 3y_n^{(iv)}y_n^{(v)}) + 15y_n^{(iv)}(4y_n'''y_n^{(v)} - 5y_n^{(iv)^2})] \quad \ldots (23)$$

Substituting b_1 in (7) and upon simplification, we get

$$a_1 = \frac{h(Vy_n + 2Uy_n')}{2U} \quad \ldots (24)$$

Substituting b_1 and b_2 in (8) and upon simplification, we get

$$a_2 = \frac{h^2[Ty_n + 14Vy_n' + 14Uy_n'']}{28U} \quad \ldots (25)$$

Substituting b_1, b_2 and b_3 in (9) and upon simplification, we get

$$a_3 = \frac{h^3[Ry_n + 3Ty_n' + 21Vy_n'' + 14Uy_n''']}{84U} \quad \ldots (26)$$

Substituting b_1, b_2, b_3 and b_4 in (10) and upon simplification, we get,

$$a_4 = \frac{h^4[Sy_n + 20Ry_n' + 30Ty_n'' + 140Vy_n''' + 70Uy_n^{(iv)}]}{1680U} \quad \ldots (27)$$

We write,

$$A = Vy_n + 2Uy_n'$$

$$B = Ty_n + 14Vy_n' + 14Uy_n''$$

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$$C = Ry_n + 3Ty_n^{'} + 21Vy_n^{''} + 14Uy_n^{'''}$$

$$D = Sy_n + 20Ry_n^{'} + 30Ty_n^{''} + 140Vy_n^{'''} + 70Uy_n^{(iv)}$$

Substituting the values of a_0, a_1, a_2, a_3, a_4, b_1, b_2, b_3 and b_4 in (5) we obtain,

$$y_{n+1} = \frac{1680Uy_n + 840Ah + 60Bh^2 + 20Ch^3 + Dh^4}{1680U + 840Vh + 60Th^2 + 20Rh^3 + Sh^4} \qquad \text{... (28)}$$

where

$U = \sum_{i=1}^{4} u_i$; $\quad S = \sum_{i=1}^{4} s_i$

$R = \sum_{i=1}^{4} r_i$; $\quad T = \sum_{i=1}^{4} t_i$

$V = \sum_{i=1}^{4} v_i$

and the u_i, s_i, r_i, t_i, v_i are as given in equations (16), (17), (19), (21) and (23) respectively and equation (28) gives the proposed method.

3. Convergence of the Method

We prove that the method is consistent and stable and hence converges.

Theorem 3.1:

The one-step, sixth-order method (28) above is consistent and convergent .

Proof:

A one-step numerical method of the form $y_{n+1} - y_n = h \varnothing (x_n, y_n ; h)$ is convergent it and only it is consistent.

We subtract y_n from both sides of (28) and we get,.

$y_{n+1} - y_n$

$$= h[1680Uy_n^{'} + 840h(Vy_n^{'} + Uy_n^{''}) + 20h^2(3Ty_n^{'} + 21Vy_n^{''} + 14Uy_n^{'''}) + 10h^3(2Ry_n^{'} + 3Ty_n^{''} + 14Vy_n^{'''} + 7Uy_n^{(4)})]$$

$1680U + 840Vh + 60Th^2 + 20Rh^3 + 3Sh^4$

$= h \varnothing (x_n, y_n, h)$, say

Hence.

$$\frac{y_{n+1} - y_n}{h} = \frac{1680Uy_n^{'} + 840h(Vy_n^{'} + Uy_n^{''}) + 20h^2(3Ty_n^{'} + 21Vy_n^{''} + 14Uy_n^{'''}) + 10h^3(2Ry_n^{'} + 3Ty_n^{''} + 14Vy_n^{'''} + 7Uy_n^{(iv)})}{1680U + 840Vh + 60Th^2 + 20Rh^3 + 3Sh^4} \qquad \text{... (29)}$$

Then

$$\lim_{h\to 0} \frac{y_{n+1} - y_n}{h} = y_n^{'} \cong f(x_n y_n) \qquad \text{... (30)}$$

Therefore, the method is consistent with the initial value problem, $y' = f(x, y)$, $y(x_0) = y_0$

Hence the method is convergent.

Theorem 3.2:

The method (28), is L-Stable .

Proof:

The method (28) is applied to the well – known Dahlquist stability test equation [4],

$y' = ky$, $y(x_0) = y_0$ and Re (k) < 0

... (31)

we have.

$U_1 = -4K^{16}y_n^4$, $U_2 = 30K^{16}y_n^4$, $U_3 = -60K^{16}y_n^4$ $U_4 = 35K^{16}y_n^4$ $U = K^{16}y_n^4$

$V_1 = 3K^{17}y_n^4$, $V_2 = -20K^{17}y_n^4$, $V_3 = 30K^{17}y_n^4$ $V_4 = -14K^{17}y_n^4$ $V = -K^{17}y_n^4$

$T_1 = -6K^{18}y_n^4$, $T_2 = 30K^{18}y_n^4$, $T_3 = 84K^{18}y_n^4$ $T_4 = -105K^{18}y_n^4$ $V = 3K^{18}y_n^4$
$R_1 = K^{19}y_n^4$, $R_2 = -42K^{19}y_n^4$, $R_3 = -30K^{19}y_n^4$ $R_4 = 70K^{19}y_n^4$ $R = -K^{19}y_n^4$
$S_1 = 56K^{20}y_n^4$ $S_2 = -10K^{20}y_n^4$ $S_3 = 60K^{20}y_n^4$ $S_4 = -105K^{20}y_n^4$ $S = 1K^{20}y_n^4$

$$y_{n+1} - y_n = \frac{hk^{17}y_n^{\,5}}{k^{16}y_n^{\,4}} \frac{\left[1680 + 0hk + 2(hk)^2 + 0(hk)^3\right]}{1680 - 840\,hk + 180(hk)^2 - 20(hk)^3 + (hk)^4} \qquad \text{... (32)}$$

$$= hky_n \frac{\left[1680 + 0\bar{h} + 2(\bar{h})^2 + 0(\bar{h})^3\right]}{1680 - 840\,\bar{h} + 180(\bar{h})^2 - 20(\bar{h})^3 + (\bar{h})^4} \qquad \text{... (33)}$$

Where $\bar{h}$ = kh

Setting S ($\bar{h}$) = $\frac{y_{n+1}}{y_n}$, we have

$$\frac{y_{n+1}}{y_n} = \frac{\bar{h}\left[1680 + 0\bar{h} + 2(\bar{h})^2 + 0(\bar{h})^3\right]}{1680 - 840\,\bar{h} + 180(\bar{h})^2 - 20(\bar{h})^3 + (\bar{h})^4} \qquad \text{... (34)}$$

is the stability function of the method (28) obviously from equation (34) as.

$$\lim_{\mathrm{Re}(\bar{h}) \to \infty} S(\bar{h}) = 0 \qquad \text{... (35)}$$

Therefore the method is L-Stable.

4. Local Truncation Error

The Local Truncation Error (LTE) is written as ,

$$O(h^9) = \left(\frac{hv}{2u}\right)\frac{y_n^{viii}}{8!} + \left(\frac{h^2T}{28U}\right)\frac{y_n^{vii}}{7!} + \left(\frac{h^3R}{84U}\right)\frac{y_n^{vi}}{6!} + \left(\frac{h^4S}{1680U}\right)\frac{y_n^{v}}{5!}$$

$$= \left(\frac{hv}{2u}\right)\frac{y_n^{viii}}{40320} + \left(\frac{h^2T}{28U}\right)\frac{y_n^{vii}}{5040} + \left(\frac{h^3R}{84U}\right)\frac{y_n^{vi}}{720} + \left(\frac{h^4S}{1680U}\right)\frac{y_n^{v}}{120} \qquad \text{... (36)}$$

5. Implementation of the Method

The efficiency of our proposed method on stiff ordinary differential equation by considering an one dimensional differential equation is given here:

Problem 1:

We consider the first order stiff IVP

y = -20y; y(0) =1

The exact solution is y= e^{-20x}

The new scheme gives accurate solution for the given problem by adaptively using the step size h which can be seen from the Table 1. The exact and new scheme solutions are plotted in logarithmic scale in Figure 1.

Table1.Results of problem 1 using our proposed method along with the exact solution

x	Exact solution	New scheme	|Error| ($\times 10^{-9}$)
0	1.0000000000	1.0000000000	0.0
0.1	0.1353352830	0.1353352861	3.1
0.2	0.0183156327	0.0183156375	4.8
0.3	0.0024787521	0.0024787507	1.4
0.4	0.0003354626	0.0003354609	1.7
0.5	0.0000453979	0.0000453999	2.0
0.6	0.0000061442	0.0000061408	3.4
0.7	0.0000008315	0.0000008292	2.3
0.8	0.0000000112	0.0000000040	7.2
0.9	0.0000000015	0.0000000004	1.1
1.0	0.0000000020	0.0000000010	1.0

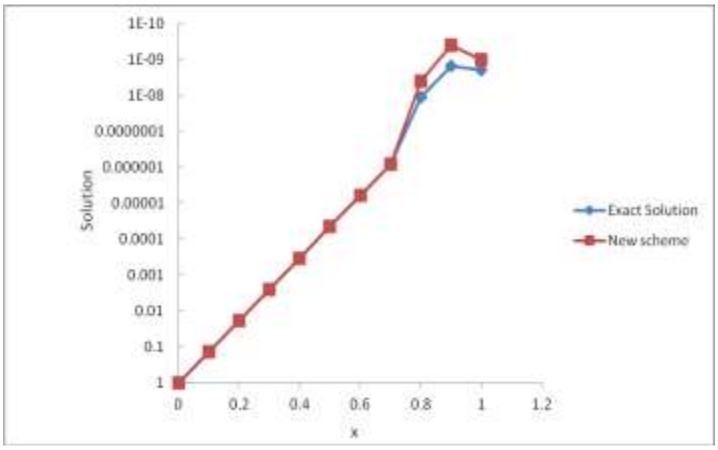

Figure 1. Results of problem 1

Problem 2:

The accuracy of our proposed method on the solution of second order differential equation $y'' + 101y' + 100y = 0$, $y(0) = 1$ and $y'(0) = -1$ is seen in the following Table 2 and Figure 2.

By setting $y' = p$, we get

$$p' = -100y - 101p$$

Here we get a 2x2 stiff system with the initial conditions $y(0) = 1$ and $p(0) = -1$ since the eigen values of the Jacobian matrix of the equation are $\lambda_1 = -1$ and $\lambda_2 = -100$. This problem is solved using our new method with $h = 0.1$ and the results by adaptively using the step size h are given

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in Table 2 below. The exact solution of the given problem is y= e^{-x}. The exact and new scheme solutions are plotted in logarithmic scale in Figure 3.

Table 2. Results of problem 2 using our proposed method along with the exact solution

x	Exact solution	New scheme	|Error| ($\times 10^{-9}$)
0	1.0000000000	1.0000000000	0.0
0.1	0.9048374180	0.9048374207	2.7
0.2	0.8187307531	0.8187307550	1.9
0.3	0.7408182207	0.7408182230	2.3
0.4	0.6703200460	0.6703200490	3.0
0.5	0.6065306597	0.6065306612	1.5
0.6	0.5488116361	0.5488116379	1.8
0.7	0.4965853070	0.4965853037	3.3
0.8	0.4493289641	0.4493289702	6.1
0.9	0.4065696597	0.4065696630	3.3
1.0	0.3678794432	0.3678794410	2.2

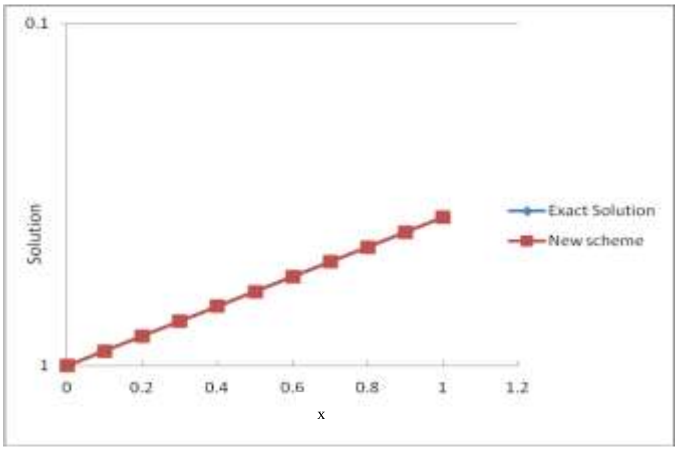

Figure 2. Results of problem 2

6. Conclusion

The theoretical analysis showed that our proposed method is a good numerical method for solving stiff initial value problems. The convergence and consistency of the proposed method is achieved. It is shown that the method is L-Stable. The LTE is obtained. The results of numerical examples seen from Tables 1 and 2 and Figures 1 and 2 show that the method gives accurate results.

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