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ANTI-MAGIC LABELING FOR BOOLEAN GRAPH OF CYCLE $BG(C_n)(n \geq 4)$

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Abstract: A graph G is anti-magic if there is a labeling of its edges with 1, 2, . . . , |E| such that the sum of the labels assigned to edges incident to distinct vertices are different. A conjecture of Hartsfield and Ringel states that every connected graph different from K_2 is anti-magic. Our main result validates this conjecture for Boolean graph of cycle $C_n(n \geq 4)$ is anti-magic.

Keywords: Boolean graph $BG(G)$, Anti-magic Labeling.

Introduction: Suppose $G = (V, E)$ is a graph. For each vertex v of G denoted by $E_G(V)$, the set of edge of G incident to v. We shall write $E(V)$ for $E_G(V)$ Let $f: E \rightarrow \{1, 2, \ldots, |E|\}$ be a bijective mapping. The vertex-sum $\varphi_f(v)$ at v is defined as $\varphi_f(v) = \sum_{e \in E(v)} f(e)$. For any two distinct vertices u, v of G, $\varphi_f(v) \neq \varphi_f(u)$ gives an anti-magic labeling of G. A graph G is called anti-magic if G has an anti-magic labeling. The problem of anti-magic labeling of graphs was introduced by Hartsfield and Ringel [4]. They conjectured that all graphs with no single edge component are anti-magic. Graph Labeling has many applications in coding theory, X-ray crystallography, radar, astronomy, circuit design, communication network addressing, and data base management.

Conjecture 1: [4]Every connected graph different from K_2 is anti-magic.

This conjecture is still open. Interestingly, the graph K_2 can be regarded as a tree on two vertices. Thus, if we restrict ourselves to trees, the above conjecture holds. Hartsfield and Ringel proved that paths, cycles and complete graph K_n, $(n \geq 3)$ are anti-magic. Recently, Alon et al. [1] have proved that the conjecture is true for some classes of dense graphs. They have shown that all dense graphs with $(n \geq 4)$ vertices and minimum degree $\Omega(\log n)$ are anti-magic. They also proved that if G is a graph with $(n \geq 4)$ vertices and the maximum degree $\Delta(G) \geq 4n - 2$, then G is anti-magic and all complete bipartite graphs except K_2 are anti-magic. Anti-magic labeling of the Cartesian product of graphs was studied in [7]; if G is a regular anti-magic graph then for any graph H, the Cartesian product $H \times G$ is anti-magic. It was proved in [4] that 2-

regular graphs are anti-magic and proved in [6] that 3-regular graphs are anti-magic. As a consequence, ifG is 2-regular or 3-regular then for any graphH, H $\times$ G is anti-magic. In this paper, we extend anti-magic labeling to Boolean Graph of cycle.

Definition 1: Boolean graphBG(G) is a graph with vertex set $V(G) \cup E(G)$ and two vertices in $BG(G)$ are adjacent if and only if they correspond to two adjacent vertices of$G$ or to a vertex and non - incident edge of G.

Theorem 1: The Boolean graph of cycle$BG(C_n), (n \geq 4)$ is anti-magic.

Proof: Let C be a cycle with the vertices $v_1, v_2, v_3, \dots, v_n$. By the definition of Boolean graph $BG(C_n)$ the vertex set is given by

$$V(BG(C_n)) = \{v_i\,; 1 \leq i \leq n\} \cup \{u_j\,; 1 \leq j \leq n\}$$

and the edge set is given by

$$E(BG(C_n)) = \{v_i\, v_{i+1}; 1 \leq i \leq n-1\} \cup \{u_j u_{j+1}\,; 1 \leq j \leq n-1\}$$

We discuss Boolean graph of cycl in two cases.

Case (a): n ≡ 0 (mod 2)

Label the vertices of $BG(C_n)$using the functionf : E $\rightarrow$ N as follows:

$f(v_i\, v_{i+1}) = i$; $i = 1, 2, \dots, n-1$ & $f(v_1, v_n) = n$

$f(u_j\, u_{j+1}) = n + j$; $j = 1, 2, \dots, n-1$ & $f(u_1, u_n) = 2n$

$f(v_i\, u_j) = (n-2)(i+1) + j + 3$ if $i < j$

and $f(v_i u_j) = (n-1)(i-1) + (n-2)j + 3$ if $i > j$

The induced function f* : V $\rightarrow$ N, such that $f^*(v) = \sum_{u \in nbd(v)} f(v_i u_j)$

We consider the when labels of vertices are distinct.

Subcase (i): when i = 1 where i< j.

$f^*(v_i) = f(v_i\, v_{i+1}) + f(v_i v_n) + \sum_{j=2}^{n-1} f(v_i u_j)$

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$$f^*(v_1) = f(v_1 v_2) + f(v_1 v_n) + \sum_{j=2}^{n-1}[(n-2)(i+1)+j+3]$$

$$f^*(v_1) = 1 + n + (n-2)(1+1)(n-2) + 3(n-2) + \frac{n(n-1)}{2} - 1$$

$$= 1 + n + 2(n-2)^2 + 3(n-2) + \frac{n(n-1)}{2} - 1$$

$$= \frac{1}{2}[2n + 2(2n^2 - n - 4n + 2) + n^2 - n]$$

$$f^*(v_1) = \frac{1}{2}[5n^2 - 9n + 4]$$

Subcase (ii): When i = 2where i< j

$$f^*(v_i) = \sum_{i=1}^{2} f(v_i v_{i+1}) + \sum_{j=3}^{n} f(v_i u_j)$$

$$= f(v_1 v_2) + f(v_2 v_3) + \sum_{j=3}^{n}[(n-2)(i+1)+j+3]$$

$$= 1 + 2 + (n-2).(n-2)(i+1) + 3(n-2) + \frac{n(n+1)}{2} - 3$$

$$f^*(v_2) = 3(n-2)^2 + 3(n-2) + \frac{n^2+n}{2}$$

$$= \frac{1}{2}[7n^2 - 17n + 12]$$

Sub case (iii): When i = 3, 4, …, n–1

$$f^*(v_i) = f(v_{i-1} v_i) + f(v_i v_{i+1}) + \sum_{\substack{j=1 \\ j \neq i-1, i}}^{n} f(v_i u_j)$$

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$$= (i-1) + i + \sum_{\substack{j=1 \\ i>j}}^{i-2} f(v_i u_j) + \sum_{\substack{j=i+1 \\ i<j}}^{n} f(v_i u_j)$$

$$= 2i-1 + \sum_{j=1}^{i-2}[(n-1)(i-1)+(n-2)j+3] + \sum_{j=i+1}^{n}[(n-2)(i+1)+j+3]$$

$$= 2i-1 + (n-1)(i-1)(i-2) + 3(i-2) + (n-2)\frac{(i-2)(i-1)}{2}$$

$$+ (n-i)(n-2)(i+1) + 3(n-i) + \frac{n(n+1)}{2} - \frac{i(i+1)}{2}$$

$$f^*(v_i) = \frac{1}{2}[(n-1)i^2 + (2n^2 - 15n + 19)i + (3n^2 + 9n - 22)]$$

Sub case (iv): When i = n

$$f^*(v_n) = f(v_1 v_n) + f(v_{n-1} v_n) + \sum_{\substack{j=1 \\ i>j}}^{n-2} f(v_i u_j)$$

$$= n + (n-1) + \sum_{j=1}^{n-2}[(n-1)(i-1)+(n-2)j+3]$$

$$= 2n - 1 + (n-1)(n-2)(i-1) + 3(n-2) + (n-2), \frac{(n-2)(n-1)}{2}$$

$$= \frac{1}{2}[(2n^2 - 6n + 4)i + n^3 - 7n^2 + 24n - 22]$$

We consider the case when labels of edges are distinct.

Subcase (v): When j = 1 where i>j

$$f^*(u_j) = f(u_j u_{j+1}) + f(u_j u_n) + \sum_{i=j+2}^{n} f(v_i u_j)$$

$$= (n+j) + 2n + \sum_{i=j+2}^{n}[(n-1)(i-1)+(n-2)j+3]$$

$$= 3n + j + (n-1)\left[\frac{n(n+1)}{2} - \frac{(j+1)(j+2)}{2}\right] + [(n-2)j - n + 4]\,(n-j-1)$$

$$= \frac{1}{2}\,[6n + 2j + (n-1)\,(n^2 + n - j^2 - 3j - 2) + 2\,(n - j - 1)\,(nj - 2j - n + 4)]$$

$$= \frac{1}{2}\,[\,(5-3n)j^2 + (2n^2 - 7n + 1)j + n^3 - 2n^2 + 13n - 6\,]$$

Subcase (vi): When j = 2, 3, …, n–2

$$f^*(u_j) = f(u_{j-1}u_j) + f(u_j\,u_{j+1}) + \sum_{\substack{i=1\\ i\neq j,\,j+1}}^{n} f(v_i u_j)$$

$$= f(u_{j-1}u_j) + f(u_j\,u_{j+1}) + \sum_{\substack{i=1\\ i<j}}^{j-1} f(v_i u_j) + \sum_{\substack{i=j+2\\ i>j}}^{n} f(v_i u_j)$$

$$= (n + j-1) + (n+j) + \sum_{i=1}^{j-1}[(n-2)(i+1) + j + 3] + \sum_{i=j+2}^{n}[(n-1)(i-1) + (n-2)\,j + 3]$$

$$= 2n + 2j - 1 + (n-2)\,\frac{(j-1)\,j}{2} + (n+j+1)\,(j-1) + (n-1)\left[\frac{n(n+1)}{2} - \frac{(j+1)(j+2)}{2}\right]$$

$$+[(n-2)j - n + 4]\,(n-j-1)$$

$$= \frac{1}{2}\,[(4n + 4j - 2) + (n-2)\,(j-1)j + 2\,(j-1)\,(n + j + 1) +$$

$$(n-1)\,[(n\,(n+1) - (j+1)\,(j+2)] + 2\,(n-j-1)\,[(n-2)j - n + 4]$$

$$= \frac{1}{2}\,[\,(5 - 2n)\,j^2 + (2n^2 - 6n + 5)\,j + n^3 - 2n^2 + 9n - 10]$$

Subcase (vii): When j = n–1 where i< j

$$f^*(u_j) = f(u_{j-1}u_j) + f(u_j\,u_{j+1}) + \sum_{\substack{i=1\\ i<j}}^{j-1} f(v_i u_j)$$

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$= (2n – 2) + (2n–1) + \sum_{i=1}^{j-1}[(n-2)(i+1)+j+3]$

$= 4n–3 +(n–2) + \frac{(j-1)j}{2} + (n+1+j) (j–1)$

$= \frac{1}{2}[nj^2 + (n+2)j+6n–8]$

Subcase (viii): When j = n where i< j

$f^* (u_j) = f (u_{j-1}u_j) + f(u_1u_j) + \sum_{\substack{i=2 \\ i<j}}^{j-1} f(v_iu_j)$

$f^* (u_j) = (n+j–1) + 2n + \sum_{i=2}^{j-1}[(n-2)(i+1)+j+3]$

$= 3n + j–1 + (n–2)\left[\frac{(j-1)j}{2}-1\right] + (n+1+j) (j–2)$

$f^* (u_j) = \frac{1}{2} [nj^2 + (n+2)j – 2]$

Case (b):n ≡ 1 (mod 2)

Let us label the vertices of $BG(C_n)$ using the function f : E → N as follows:

$f(v_i\, v_{i+1}) = 2i-1, i = 1, 2, …, n–1$

$f(v_1v_n) = 2n – 1$

$f(u_j\, u_{j+1}) = 2j ; j = 1, 2, …, n–1$

$f(u_1\, u_n) = 2n$

$f(v_iu_j) = (n–2) (i+1) + j + 3$ if i< j

and $f (v_iu_j) = (n–1) (i–1)+ (n-2)j+3$ for i> j

The induced function f* : V → N such that $f^* (v) = \sum_{u \in nbd(v)} f(v_iu_j)$

We consider the when the labels are distinct.

Subcase (i): When i = 1 where i< j

$$f^*(v_i) = f(v_i v_{i+1}) + f(v_i v_n) + \sum_{\substack{j=2 \\ i<j}}^{n-1} f(v_i u_j)$$

$$= (2i-1) + (2n-1) + \sum_{j=2}^{n-1}[(n-2)(i+1)+j+3]$$

$$= (2i–1) + (2n–1) + [(n–2)(i+1) + 3](n–2) + \frac{(n-1).n}{2} - 1$$

$$= \frac{1}{2}[(2n^2 – 8n + 12)i + 3n^2 + n – 10]$$

Subcase (ii): When i = 2 where i< j

$$f^*(v_i) = f(v_{i-1} v_i) + f(v_i v_{i+1}) + \sum_{\substack{j=3 \\ i<j}}^{n} f(v_i u_j)$$

$$= 2(i–1) – 1 + 2i – 1 + \sum_{j=3}^{n}[(n-2)(i+1)+j+3]$$

$$= 4i–4 + [(n-2)(i+1) + 3](n-2) + \left[\frac{n(n+1)}{2} - 1 - 2\right]$$

$$= \frac{1}{2}[8i – 8 + (2n–4)(ni + n – 2i + 1) + (n^2 + n) – 6]$$

$$= \frac{1}{2}[8i – 8 + 2n^2 i + 2n^2 – 4ni + 2n – 4ni – 4n + 8i – 4 + n^2 + n – 6]$$

$$= \frac{1}{2}[(2n^2 – 8n + 16)i + 3n^2 – n – 18]$$

Subcase (iii): When i = 3, 4, 5, …, n–1

$$f^*(v_i) = f(v_{i-1} v_i) + f(v_i v_{i+1}) + \sum_{\substack{j=1 \\ j\neq i-1,i}}^{n} f(v_i u_j)$$

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$$= 2(i–1)–1 + 2i – 1 + \sum_{\substack{j=1 \\ i>j}}^{i-2} f(v_i u_j) + \sum_{\substack{j=i+1 \\ i<j}}^{n} f(v_i u_j)$$

$$= 4i – 4 + \sum_{j=1}^{i-2}[(n-1)(i-1)+(n-2)j+3] + \sum_{j=i+1}^{n}[(n-2)(i+1)+j+3]$$

$$= 4i – 4 + [(n–1)(i–1)+3](i–2) + (n–2)\frac{(i-2)(i-1)}{2} + [(n–2)(i+1)+3](n-i) + \left[\frac{n(n+1)}{2}-\frac{i(i+1)}{2}\right]$$

$$= \frac{1}{2}[8i – 8 + (2i – 4)(ni – n – i+4) + (n–2)(i^2 – 3i + 2) + (ni + n – 2i + 1)(2n – 2i) + n^2 +n – i^2 – i)$$

$$f^*(v_i) = \frac{1}{2}[(n-1)i^2 + (2n^2 – 15n + 23)i+3n^2 + 9n – 28].$$

Subcase (iv): When $i = n$

$$f^*(v_i) = f(v_{i-1} v_i) + f(v_1 v_i) + \sum_{\substack{j=1 \\ i>j}}^{n-2} f(v_i u_j)$$

$$= 2(i–1) – 1 + 2n – 1 + \sum_{j=1}^{n-2}[(n-1)(i-1)+(n-2)j+3]$$

$$= 2i – 3 + 2n – 1 + [(n–1)(i–1) + 3](n–2) + (n–2)\left[\frac{(n-2)(n-1)}{2}\right]$$

$$= \frac{1}{2}[(4n + 4i – 8) + (2n – 4)[ni – n – i + 4] + (n–2)(n^2 – 3n + 2)$$

$$= \frac{1}{2}[(2n^2 – 6n + 8)i+n^3 – 7n^2 + 24n – 28].$$

We consider the case when the labels of edges are distinct.

Sub case (v): When j = 1 where i> j.

$$f^*(u_j) = f(u_j u_{j+1}) + f(u_j u_n) + \sum_{\substack{i=j+2 \\ i>j}}^{n} f(v_i u_j)$$

$$= 2j + 2n + \sum_{i=j+2}^{n} [(n-1)(i-1)+(n-2)j+3]$$

$$= 2j + 2n + (n-1)\left[\frac{n(n+1)}{2} - \frac{(j+1)(j+2)}{2}\right] + [(n-2)j - n + 4]\,(n-j-1)]$$

$$= \frac{1}{2}[4j + 4n + (n-1)(n^2 + n - j^2 - 3j - 2) + 2(n-j-1)(nj - 2j - n + 4)]$$

$$f^*(u_j) = \frac{1}{2}[(5-3n)j^2 + (2n^2 - 7n + 3)j + n^3 - 2n^2 + 11n - 6]$$

Subcase (vi): When j = 2, 3, …, n-2

$$f^*(u_j) = f(u_{j-1}u_j) + f(u_j u_{j+1}) + \sum_{\substack{i=1 \\ i\neq j,\, j+1}}^{n} f(v_i u_j)$$

$$= f(u_{j-1} u_j) + f(u_j u_{j+1}) + \sum_{\substack{i=1 \\ i<j}}^{j-1} f(v_i u_j) + \sum_{i=j+2}^{n} f(v_i u_j)$$

$$= 2(j-1) + 2j + \sum_{i=1}^{j-1}[(n-2)(i+1)+j+3] + \sum_{i=j+2}^{n}[(n-1)(i-1)+(n-2)j+3]$$

$$= 2j - 2 + 2j + (n-2)\frac{(j-1)j}{2} + (n+j+1)(j-1) + (n-1)\left[\frac{n(n+1)}{2} - \frac{(j+1)(j+2)}{2}\right] + [(n-2)j - n + 4]\,(n-j-1)$$

$$= \frac{1}{2}[8j - 4 + (n-2)(j^2 - j) + 2(j-1)(n+j+1) + (n-1)(n^2 + n - j^2 - 3j - 2) + 2(n-j-1)(nj - 2j - n + 4]$$

$$= \frac{1}{2}[(5-2n)j^2 + (2n^2 - 6n + 9)j + n^3 - 2n^2 + 5n - 12].$$

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Subcase (vii): When j = n–1 where i< j

$$f^*(u_j) = f(u_{j-1}u_j) + f(u_j u_{j+1}) + \sum_{\substack{i=1 \\ i<j}}^{j-1} f(v_i u_j)$$

$$= 2(j-1) + 2j + \sum_{i=1}^{j-1}[(n-2)(i+1) + j + 3]$$

$$= 4j - 2 + (n-2)\frac{(j-1)j}{2} + (n+1+j)(j-1)$$

$$= \frac{1}{2}[8j - 4 + (n-2)j(j-1) + 2(j-1)(n+1+j)]$$

$$= \frac{1}{2}[nj^2 + (n+10)j - 2n - 6].$$

Subcase (viii): When j = n where i< j

$$f^*(u_j) = f(u_{j-1}u_j) + f(u_1u_j) + \sum_{\substack{i=2 \\ i<j}}^{j-1} f(v_i u_j)$$

$$= 2(j-1) + 2n + \sum_{i=2}^{j-1}[(n-2)(i+1) + j + 3]$$

$$= 2n + 2j - 2 + (n-2)\frac{(j-1)j}{2} - 1 + (n+j+1)(j-2)$$

$$= \frac{1}{2}[4n + 4j - 4 + (n-2)(j^2 - j - 2) + 2(j-2)(n+j+1)$$

$$= \frac{1}{2}[nj^2 + (n+4)j - 2n - 4]$$

As a whole the labeling of all the vertices and the edges of the Boolean graph of cycle is anti-magic.

$\therefore BG(C_n)$ is anti-magic.

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