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VERTEX TRIANGULAR GRAPHS

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Abstract

A (p, q) graph is said to admit vertex triangular labeling if its vertices can be labeled by the first p triangular numbers such that the edge labels obtained by adding the labels of its end vertices are distinct. A graph which admits a vertex triangular labeling is called a vertex triangular graph. In this paper we discuss some properties of vertex triangular labeling and we give some classes of graphs which are vertex triangular.

Keywords:

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1. Introduction

For all terminology and notation in graph theory, not specifically defined in this paper we refer the reader to the text-book by Harary[1] and for number theory we follow Burton[2]. Unless mentioned otherwise, all graphs in this paper are finite, simple and undirected.

A graph labeling is an assignment of integers, real numbers, sets etc to the vertices or edges or both subject to certain conditions. Most of the labeling problems have three important ingredients. A set of numbers from which vertex labels are chosen, a rule that assigns a value to each edge, a condition that these values must satisfy. Vast amount of literature about different types of graph labeling is available and more than 1000 research papers have been published so far in last four decades. A dynamic survey of graph labeling is regularly updated by Gallian[3] and is published by Electronic Journal of Combinatory. In this paper we introduce and study a new type of graph labeling which we call the Vertex Triangular Graph labeling.

Definition 1.1[2]

A number is triangular if and only if it is of the form $\frac{n(n+1)}{2}, n \geq 1$
If the n^{th} triangular number is denoted by T_n then $T_n = 1 + 2 + 3 + \cdots + n$
The triangular numbers are 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78…… We denote the set of first p triangular numbers by $\Delta_{\mathbf{p}}$. That is $\Delta_{\mathbf{p}} = \{T_1, T_2, \ldots \ldots T_p\}$

Result 1.2[2]

The sum of n triangular numbers is $\frac{n(n+1)(n+2)}{6}, n \geq 1$

2. VERTEX TRIANGULAR GRAPHS

Definition 2.1

A vertex triangular labeling of a graph $G = (p, q)$ is a bijective function $f: V(G) \to \Delta_{\mathbf{p}}$ such that the induced function f^+: $E(G) \to \{4,5, \ldots . p^2\}$ defined by $f^+(uv) = f(u) + f(v)$ is injective. A graph G is said to be a Vertex Triangular Graph if it admits a Vertex Triangular labeling.

Observation 2.2
For any (p, q) Vertex Triangular Graph $4 \leq f^+(e) \leq T_p + T_{p-1}$.

PROPERTIES OF VERTEX TRIANGULAR GRAPHS

Theorem 2.3
If $G = (p, q)$ is a Vertex Triangular $k - regular$ graph then Σf^+ (e)= $\frac{kp(p+1)(p+2)}{6}$

Proof
Since G is $k - regular$, each vertex is counted k times in Σf^+ (e)
That is Σf^+ (e)= $k(T_1 + T_2 + \cdots T_p)$ = $\frac{kp(p+1)(p+2)}{6}$ by result 1.2 ■

Illustration 2.4
1. For C_5, Σf^+ (e) $= \frac{2.5.6.7}{6} = 70$
2. For K_4, Σf^+ (e) $= \frac{3.4.5.6}{6} = 60$

Theorem 2.5

In any Vertex Triangular Graph we cannot label T_{3n-2} & T_{6n-1} to adjacent vertices if T_{3n} & T_{6n-2} are labels of adjacent vertices $n = 1,2, \ldots \ldots$

Proof

Let G be a Vertex Triangular Graph. Let u & v are two adjacent vertices with labels $f(u) = T_{3n}$ & $f(v) = T_{6n-2}$

Let $e_1 = (uv)$

Then $f^+(e_1) = f(u) + f(v) = \frac{3n(3n+1)}{2} + \frac{(6n-2)(6n-1)}{2} = \frac{45\,n^2-15n+2}{2}$, $n = 1,2, \dots\dots$

Now if u^1 and v^1 are also adjacent vertices with $f(u^1) = T_{3n-2}$ and $f(v^1) = T_{6n-1}$

$f^+(e^1) = f(u^1) + f(v^1) = \frac{(3n-2)(3n-1)}{2} + \frac{(6n-1)(6n)}{2} = \frac{45\,n^2-15n+2}{2}$, $n = 1,2, \dots\dots$

Where $e^1 = (u^1v^1)$

That is $f^+(e_1) = f^+(e^1)$, not possible. ■

Theorem 2.6

Let $G = (p, q)$ be a Vertex Triangular Graph with a Vertex Triangular labeling f. If $f^+(e) \equiv 1(mod\ 2)\ \forall e \in E(G)$ then G is bipartite.

Proof

Suppose $f^+(e) \equiv 1(mod\ 2)\ \forall e \in E(G)$

Let $e \in E(G)$, then $f^+(e) = T_i + T_j$ for some $T_i , T_j \in \Delta_p$

and $T_i + T_j \equiv 1(\mathrm{mod} 2)$

Since $T_i + T_j$ is an odd number, we have T_i & T_j are of opposite parity.

Let $X = \{u \in V(G) : f(u) is\ even\}$ & $Y = \{v \in V(G) : f(v) is\ odd\}$

Then the sets X & Y form a disjoint partition of $V(G)$ and any edge with $f^+(e) \equiv 1(mod\ 2)$ is of one end in X and the other end in Y. ■

Corollary 2.7

If G is a graph with $f^+(e) \equiv 1(mod\ 2n)\ \forall e \in E(G)$ then G is bipartite.

Proof

If $f^+(e) \equiv 1(mod\ 2n)\ \forall e \in E(G) \Rightarrow f^+(e) \equiv 1(mod\ 2)\ \forall e \in E(G)$

Then by theorem 2.6, G is bipartite. ■

Remark 2.8

Every spanning subgraph H of a Vertex Triangular Graph G is Vertex Triangular, since $V(H) = V(G)$ & $f\big(V(H)\big) = f\big(V(G)\big) = \{ T_1, T_2, T_3, \dots, T_p \}$. In general subgraphs of a Vertex Triangular Graph need not be Vertex Triangular. For example K_5 with one pendant vertex is vertex triangular but K_5 is not vertex triangular. Hence Vertex Triangular labeling is not a hereditary property.

CLASSES OF VERTEX TRIANGULAR GRAPHS

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Theorem 2.9

Every path P_n is a Vertex Triangular Graph.

Proof

Let V(P_n)={$v_1, v_2, v_3 \ \ldots\ldots, v_n$ } where v_1 *and* v_n are the end vertices with
v_i adjacent to v_{i+1} for $1 \le$ i $\le$ n-1
Define $f: V(P_n) \to \Delta_{\mathbf{p}}$ by $f(v_i) = T_i$, i=1,2,3,….n
Since f is increasing on the vertex set and
$f^+(v_i v_{i+1}) = T_i + T_{i+1}$, f^+ is also an increasing function on the edge set. Hence
f^+ is injective and f is a Vertex Triangular labeling on P_n. ■

Theorem 2.10

The cycle C_n is a Vertex Triangular Graph $\forall n$.

Proof

Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of the cycle C_n
Case 1: $\boldsymbol{n \neq 5}$.
Define $f: V(C_n) \to \Delta_n$ by $f(v_i) = T_i$, i=1, 2, 3,…n
Then
$f^+(v_i\ v_{i+1}) = (i+1)^2$, i=1, 2, 3….n-1
Also $f^+(v_1\ v_n) = f(v_1) + f(v_n) = 1 + \frac{n(n+1)}{2}$
Since
$f^+(v_1\ v_n)$ is not a perfect square and $f^+$is increasing on $(v_i\ v_{i+1}), i = 1,2,3, \ldots, n-1$,
f^+ is injective and hence C_n is a Vertex Triangular Graph .
***Case* 2: $\boldsymbol{n = 5}$**
When $\boldsymbol{n = 5}$ the above labeling is not possible, since $f^+(e_5) = f^+(e_3)$ and f^+ is not injective. So we label $C_5: v_1, v_2, \ldots, v_5$ as follows,
$f(v_1) = 1, f(v_2) = 6, f(v_3) = 3, f(v_4) = 10, f(v_5) = 15$ ■

Theorem 2.11

The complete graph K_n is a Vertex Triangular Graph if and only if $n \le 4$

Proof

The Vertex Triangular labeling of K_n for $n \le 4$ are as follows. $f(v_i) = T_j, i, j = 1,2,3,4$ where v_i are the vertices of K_n. Then $f^+$is injective and hence K_n is a Vertex Triangular Graph. If $n \ge 5$, $T_3 + T_4 = T_1 + T_5$, So $f^+$is not injective. So K_n, $n \ge 5$ is not a Vertex Triangular Graph. Therefore the complete graph K_n is a Vertex Triangular Graph if and only if $n \le 4$. ■

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Theorem 2.12

Every star graph $K_{1,n}$ is a Vertex Triangular Graph.

Proof

Let u be the apex vertex and $v_1, v_2, v_{3 \ldots\ldots}, v_n$ be the pendant vertices of the star graph $K_{1,n}$.
Define, $f(u) = 1$ and $f(v_i) = T_{i+1}$, $i = 1,2,3 \ldots., n$
Then the edge labels are, $f^+(uv_i) = 1 + T_{i+1} = 1 + \frac{(i+1)(i+2)}{2}$
Since f is increasing on $V(G)$, f^+ is also increasing. So f^+ is injective.
Hence $K_{1,n}$ is a Vertex Triangular Graph. ■

Theorem 2.13

The graph $G = K_2 + m\,K_1$ is a Vertex Triangular Graph.

Proof

Let $V(G) = \{v_1, v_2, v_3, \ldots, v_{m+2}\}$ where $V(K_2) = \{v_1, v_2\}$
define $f: V(G) \to \{T_1, T_2, T_3, \ldots., T_{m+2}\}$ by $f(v_i) = T_i$, for i=1,2,3,….m+2
The induced function is injective, for if $f^+(v_1 v_i) = f^+(v_2 v_j)$ where $i, j \neq 1,2$
By taking, $f(v_i) = x$ & $f(v_j) = y$, we get an equation of the form $x - y = 2$ then x = 3 & $y = 1$, which is not possible by definition. Hence G is a Vertex Triangular Graph.
■

Theorem 2.14

Every wheel graph $W_n = K_1 + C_{n-1}$ is a Vertex Triangular Graph.

Proof

Let v be the apex vertex and $v_1, v_2, \ldots., v_{n-1}$ be the other vertices.
Define $f(v) = 1$
Then label each v_i with T_{i+1} in the order that 6 & 10 will not be given to adjacent vertices if $n \geq 4$. ■

Theorem 2.15

The ladder L_n is a Vertex Triangular Graph.

Proof

Let $V(L_n) = \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots.., b_n\}$ and
$E(L_n) = \{a_i b_i \,/\, 1 \leq i \leq n\} \cup \{a_i a_{i+1} \,/\, 1 \leq i \leq n-1\} \cup \{b_i b_{i+1} / \, 1 \leq i \leq n-1\}$
Define $f: V(L_n) \to \Delta_{2n}$ as follows
$f(a_i) = T_{2i-1}, \; 1 \leq i \leq n$
$f(b_i) = T_{2i}, \; 1 \leq i \leq n$ so that

$f^+(a_ib_i) = 4i^2, \ 1 \le i \le n$
$f^+(a_ia_{i+1}) = 4i^2 + 2i + 1, \quad i = 1,2, \ldots n-1$
$f^+(b_ib_{i+1}) = 4i^2 + 6i + 3, \ i = 1,2, \ldots n-1$
Since $f^+(a_ib_i)$ is an even number for $1 \le i \le n$ and $f^+(a_ia_{i+1})$ & $f^+(b_ib_{i+1})$ are odd number for $1 \le i \le n-1$,
$f^+(a_ib_i) \neq f^+(a_ia_{i+1})$
$f^+(a_ib_i) \neq f^+(b_ib_{i+1})$
Also f is an increasing function on $V(G)$ and so $f^+(a_{i,}b_i) \neq f^+(a_j b_j) \ \forall i \neq j$
And $f^+(a_ia_{i+1}) \neq f^+(a_j a_{j+1}), \ i \neq j, \ 1 \le i \le n-1, \ 1 \le j \le n-1$
Also and $f^+(b_ib_{i+1}) \neq f^+(b_jb_{j+1}), 1 \le i, j \le n-1$, since these numbers form an increasing sequence of odd numbers.
Hence f^+ is injective and f is a Vertex Triangular labeling. ■

Theorem 2.16

The complete bipartite graph $K_{2,n}$ is a vertex triangular graph for any n.

Proof
Let $X = \{u_1, u_2\}$ & $Y = \{v_1, v_2, \ldots, v_n\}$ be the bipartition of the vertices of $K_{2,n}$.
Define $f(u_1) = 1$, $f(u_2) = 3$ and $f(v_i) = T_{i+2}, i = 1,2, \ldots, n$
Then the edge labels are $\forall i$,

$$f^+(u_1v_i) = 1 + T_{i+2} = \frac{(i+2)(i+3)}{2} + 1 = \frac{i^2 + 5i + 8}{2}$$
$$f^+(u_2v_i) = 3 + T_{i+2} = \frac{(i+2)(i+3)}{2} + 3 = \frac{i^2 + 5i + 12}{2}$$

If $f^+(u_1v_i) = f^+(u_2v_j)$, for some i, j.
Then,

$$\frac{i^2 + 5i + 8}{2} = \frac{j^2 + 5j + 12}{2}$$
$$j^2 - i^2 + 5(j - i) + (12 - 8) = 0$$
$$(i - j)(i + j + 5) = 6 \qquad \ldots\ldots (1)$$

Since i & j are positive integers (1) is true only if
$(i - j) = 1$ and $i + j + 5 = 6$ or $(i - j) = 2$ and $i + j + 5 = 3$
If $(i - j) = 1$ and $i + j + 5 = 6$, then $i = 1$ *and* $j = 0$, not possible.
If $(i - j) = 2$ and $i + j + 5 = 3$ then, $i = 0$ *and* $j = -2$, not possible,
Therefore f^+ is injective.

Therefore the complete bipartite graph $K_{2,n}$ is a vertex triangular graph for any n. ■

Theorem 2.17

The graphs $K_{m,m}, 1 \le m \le 6$ are Vertex Triangular Graphs.

Proof

Let $X = \{u_1, u_2, \ldots u_m\}$ & $Y = \{v_1, v_2, \ldots v_m\}$ be the bipartition of $K_{m,m}, 1 \le m \le 6$. Define $f(u_i) = T_{2i-1}$ & $f(v_i) = T_{2i}$. Suppose if possible $e_{ij} = (u_i v_j)$ & $e_{mn} = (u_m v_n)$ are two different edges with $f^+(e_{ij}) = f^+(e_{mn})$

Then, $\dfrac{(2i-1)(2i)+(2j)(2j+1)}{2} = \dfrac{(2m-1)(2m)+(2n)(2n+1)}{2}$

That is, $2i^2 - i + 2j^2 + j = 2m^2 - m + 2n^2 + n$

That is, $2(i^2 + j^2 - m^2 - n^2) = i - j - m + n$

So $2(i^2 - m^2 + j^2 - n^2) = (i-m) - (j-n)$ ....(1)

(1) Is true only if both sides of (1) are zero. That is $i = m$ & $j = n$, a contradiction to the choice of $e_{ij} = (u_i v_j)$ & $e_{mn} = (u_m v_n)$. Hence f^+ is injective and $K_{m,m}, 1 \le m \le 6$ are vertex triangular. ■

Theorem 2.18

The comb $P_n \odot K_1$ admits a Vertex Triangular labeling.

Proof

Let $P_n: u_1, u_2, \ldots, u_n$ be the path and let $w_i = u_i u_{i+1} (1 \le i \le n-1)$ be the edges. Let $v_1, v_2, \ldots, v_n$ be the pendant vertices adjacent to $u_1, u_2, \ldots, u_n$ respectively and let $t_i = u_i v_i (1 \le i \le n)$ be the edges.

For $i = 1,2, \ldots n$, define $f(u_i) = T_{2i}$ and $f(v_i) = T_{2i-1}$.

Then, $f^+(w_i) = f(u_i) + f(u_{i+1}) = T_{2i} + T_{2(i+1)} = 4i^2 + 6i + 3$, an odd number and $f^+(t_i) = f(u_i) + f(v_i) = T_{2i} + T_{2i-1} = 4i^2 = (2i)^2$, an even number and hence $f^+(w_i) \ne f^+(t_j), \forall i, j$

Also $f^+(t_i) \ne f^+(t_j)$ and $f^+(w_i) \ne f^+(w_j), \forall i \ne j$.

Therefore f^+ is injective and f is a Vertex Triangular labeling. ■

Theorem 2.19

Bistar graph $B_{m,n}$ is a Vertex Triangular Graph $\forall m, n$.

Proof

Let,

$V(B_{m,n}) = \{u, v, u_i, v_j, 1 \le i \le m, 1 \le j \le n\}$

$E(B_{m,n}) = \{uv, uu_i, vv_j, 1 \le i \le m, 1 \le j \le n\}$

Define $f(u) = 1$, $f(v) = 3$, $f(u_i) = T_{i+2}, i = 1, 2, \ldots, m$ and $f(v_j) = T_{m+2+j}, j = 1, 2, .., n$

Then $f^+(uv) \neq f^+(uu_i) \neq f^+(vv_j), \forall i, j$. Therefore the induced edge labels are distinct and hence Bistar graph $B_{m,n}$ is a Vertex Triangular Graph $\forall m, n$. ■

Conclusion

In this paper we introduced a new concept of vertex triangular graphs using triangular numbers and state some properties of vertex triangular graphs. We identified many classes of graphs which are vertex triangular.

References

[1] F Harary, *Graph Theory*, Narosa Publishing House, 2001

[2] David M Burton, *Elementary Number Theory*, Sixth edition, Tata McGraw-Hill Publishing Company Limited, New Delhi

[3] J A Gallian, *A Dynamic survey of Graph labeling*, Electronic Journal of Combiatorics,17(2010),DS6

[4] S Arumugam, K A Germina and Ajitha V, *On Square Sum Graphs* AKCE International Journal of Graphs and Combinatorics, 1-10(2008).

[5] S. Murugesan, D. Jayaraman and J. Shiama, *Second and Third order triangular sum labeling of graphs*, International Journal of Mathematical Archive-4(2), Feb-2013, 55-62, ISSN 2229- 5046.

[6] S.M. Hegde and P.Shankaran , *On triangular sum labeling of graphs, in Labeling of Discrete Structures and Applications*, Narosa Publishing House, New Delhi, 2008, 109- 115.

[7] S. Uma Maheswari, Dr. K. Indirani, *On Triangular Sum Prime Graphs*, International Journal of Computer & Mathematical Sciences, ISSN 2347 – 852 Volume 5, Issue 3, 2016

[8] S.K Vaidya, U. M.Prajapati and P.L. Vihol, *Some Important Results on Triangular Sum Graphs*, applied mathematical Sciences, Vol. 3,2009, no.36, 1763-1772