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A NEW SUBCLASS OF MEROMORPHIC FUNCTIONS INVOLVING THE POLYLOGARITHM FUNCTION

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Abstract

In this paper we introduce and study a new subclass of meromorphic function with alternating coefficients involving the Polylogaitham function defined by a new operator $D_c f(z)$ and obtain coefficients estimates, growth and distortation and integral transforms.

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1. Introduction

Historically, the classical polylogarithm function was invented in 1696, by Leibniz and Bernoulli, as mentioned in [3]. For |z| < 1 and c a natural number with $c \geq 2$, the polylogarithm function (which is also known as Jonquiere's function) is defined by the absolutely convergent series :

$$Li_c(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^c} \tag{1.1}$$

Later on, many mathematicians studied the polylogarithm function such as Euler, Spence, Abel,

Lobachevsky, Rogers, Ramanujan, and many others [4], where they discovered many functional identities by using polylogarithm function. However, the work employing polylogartihm has been stopped many decades later. During the past four decades, the work using polylogarithm has again been intensified vividly due to its importance in many fields of mathematics, such as complex analysis, algebra, geometry, topology, and mathematical physics (quantum field theory) [5-7]. In 1996, Ponnusamy and Sabapathy discussed the geometric mapping properties of the generalized polylogarithm [8]. Recently, Al-Shaqsi and Darus generalized Ruscheweyh and Salageean operators, using polylogarithm functions on class *A* of analytic functions in the open unit dist U = {z : |z| < 1}. By making use of the generalized operator they introduced certain new subclasses of *A* and investigated many related properties [9]. A year later, same authors again employed the nth order polylogarithm function to define a multiplier transformation on the class *A* in U [10].

To the best of our knowledge, no research work has discussed the polylogarithm function in conjunction with meromorphic functions. Thus, in this present paper, we redefine the polylogarithm function to be on meromorphic type.

Let Σ denote the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{k=0}^{\infty}a_k z^k \,, \tag{1.2}$$

Which are analytic in the punctured open unit disk

$$U^{\bullet}:=\{z:z\in C, 0<|z|<1\}=:U\setminus\{0\} \tag{1.3}$$

A function $f(z)$ in Σ is said to be meromorphically starlike of order δ if and only if

$$\Re\left\{-\frac{zf'(z)}{f(z)}\right\}>\delta;\,(z\in U^{\bullet}), \tag{1.4}$$

for some $\delta(0\le\delta<1)$. We denote by $\Sigma^{\bullet}(\delta)$ the class of all meromorophically starlike order δ. Furthermore, a function $f(z)$ in Σ is said to be meromorphically convex of order δ if and only if

$$\Re\left\{-1-\frac{zf''(z)}{f'(z)}\right\}>\delta;\,(z\in U^{\bullet}), \tag{1.5}$$

for some $\delta(0\le\delta<1)$. We denote $\Sigma_K(\delta)$ the class of all meromorphically convex order δ. For functions $f\in\Sigma$ given by (1.2) and $g\in\Sigma$ given by

$$g(z)=\frac{1}{z}+\sum_{k=0}^{\infty}b_k z^k \,, \tag{1.6}$$

we define the Hadamard product (or convolution) of f and g by

$$(f*g)(z)=\frac{1}{z}+\sum_{k=0}^{\infty}a_k b_k z^k \tag{1.7}$$

Let Σ_p be the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k ;\ a_k \geq 0 \tag{1.8}$$

which are analytic and univalent in $U^{\bullet}$

Liu and Srivastava [11] defined a function $h_p(\alpha_1,.......,\alpha_q;\beta_1,....,\beta_s;z)$ by multiplying the well known generalized hypergeometric function ${}_qF_s$, with z^{-p} as follows:

$$h_p(\alpha_1,.......,\alpha_q;\beta_1,....,\beta_s;z) \tag{1.9}$$

$$= z^{-p}\,{}_qF_s(\alpha_1,.......,\alpha_q;\beta_1,....,\beta_s;z)$$

where $\alpha_1,.......,\alpha_q;\beta_1,....,\beta_s$ are complex parameters and q ≤ s + 1, $p \in N$

Analogous to Liu and Srivastava work [9] and corresponding to a function $\Phi_c(z)$ given by

$$\Phi_c(z) = z^{-2} Li_c(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \frac{1}{(k+2)^c} z^k \tag{1.10}$$

we consider a linear operator $\Omega_c f(z) : \Sigma \to \Sigma$ which is defined by the following Hadamard product (or convolution):

$$\Omega_c f(z) = \Phi_c(z) * f(z) \tag{1.11}$$

$$= \frac{1}{z} + \sum_{k=0}^{\infty} \frac{1}{(k+2)^c} a_k z^k$$

Next, we define the linear operator $D_c f(z) : \Sigma \to \Sigma$ as follows :

$$D_c f(z) = \left\{ \Omega_c f(z) - \frac{1}{2^c} a_0 \right\}$$

$$= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{1}{(k+2)^c} a_k z^k \tag{1.12}$$

Let $M_c(\alpha,\beta,\gamma)$ denote the class of function $f(z)$ in Σ satisfying the condition

$$\left| \frac{z(D_c f(z))' + 1}{(2\gamma - 1) z^2 (D_c f(z))' + (2\alpha\gamma - 1)} \right| < \beta$$

for some $\alpha(0 \leq \alpha < 1), \beta(0 < \beta \leq 1), \gamma(\frac{1}{2} \leq \gamma \leq 1)$,

and for all $z \in U$. We note that $M_0(\alpha,\beta,\gamma) = \Sigma(\alpha,\beta,\gamma)$

Let σ_A be the subclass of Σ which consisting of function of the form

$$f(z) = \frac{1}{z} + a_0 z - a_1 z^2 + a_3 z^3 - ....,\quad a_k \geq 0,$$

$$= \frac{1}{z} + \sum_{k=1}^{\infty} (-1)^{k-1} a_k z^k, a_k \geq 0 \tag{1.14}$$

and let $\sigma^*_{A,C}(\alpha,\beta,\gamma) = M_c(\alpha,\beta,\gamma) \cap \sigma_A$

In this paper, coefficient inequalities and distortion theorem for the class $\sigma^*_{A,C}(\alpha,\beta,\gamma)$ are determine. Techniques used are similar to these of Siliverman [13], Uralegaddi and Ganigi [14], Aouf and Darwish [1] and Aouf and Hossen [2]. Finally, the class preserving integral operators of the form

$$F(z) = \frac{s}{z^{s+1}} \int_0^z k^s f(t)dt \ (s > 0) \tag{1.15}$$

is considered .

2. Coefficient Inequalities

Theorem 2.1. Let $f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k$, if

$$\sum_{k=1}^{\infty} k(k+2)^{-c}(1+2\beta\gamma-\beta)|a_k| \leq 2\beta\gamma(1-\alpha) \tag{2.1}$$

Then $f(z) \in \sigma^*_{A,c}(\alpha,\beta,\gamma)$.

Proof. Suppose (2.1) holds for all and admissible values of α,β,γ and c It suffices to show that

$$\left| \frac{z^2(D_c f(z))' + 1}{(2\gamma-1)z^2(D_c f(z))' + (2\alpha\gamma-1)} \right| < \beta \tag{2.2}$$

for |z| < 1. We have

$$\left| \frac{z^2(D_c f(z))' + 1}{(2\gamma-1)z^2(D_c f(z))' + (2\alpha\gamma-1)} \right| = \left| \frac{\sum_{k=1}^{\infty} k(k+2)^{-c} a_k z^{k+1}}{2\gamma(1-\alpha) - \sum_{k=1}^{\infty}(2\gamma-1)k(k+2)^{-c} a_k z^{k+1}} \right|$$

$$\leq \frac{\sum_{k=1}^{\infty} k(k+2)^{-c} |a_k|}{2\gamma(1-\alpha) - \sum_{k=1}^{\infty}(2\gamma-1)k(k+2)^{-c}|a_k|}$$

The last expression is bounded above by β, provided

$$\sum_{k=1}^{\infty} k(k+2)^{-c} |a_k| \le \beta\left\{2\gamma(1-\alpha) - \sum_{k=1}^{\infty}(2\gamma-1)k(k+2)^{-c}|a_k|\right\}$$

which is equivalent to

$$\sum_{k=1}^{\infty} k(k+2)^{-c}(1+2\beta\gamma-\beta)|a_k| \le 2\beta\gamma(1-\alpha) \qquad (2.3)$$

which is true by hypothesis

Theorem 2.2. A function $f(z)\, in\, \sigma_A$ is in $\sigma^*_{A,c}(\alpha,\beta,\gamma)$ if and only if

$$\sum_{k=1}^{\infty} k(k+2)^{-c}(1+2\beta\gamma-\gamma)a_k \le 2\beta\gamma(1-\alpha).$$

Proof. In view of Theorem 2.1 it suffices to show that the only if part. Let us assume that

$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty}(-1)^{k-1}a_k z^k \quad (a_k \ge 0)$ is in $\sigma^*_{A,C}(\alpha,\beta,\gamma)$. Then

$$\left|\frac{z^2(D_c f(z)'+1}{(2\gamma-1)z^k(D_c f(z))'+(2\alpha\gamma-1)}\right| = \left|\frac{\sum_{k=1}^{\infty}(-1)^{k-1}k(k+2)^{-c}a_k z^{k+1}}{2\gamma(1-\alpha)-\sum_{k=1}^{\infty}(-1)^{k-1}(2\gamma-1)k(k+2)^{c}a_k z^{k+1}}\right| < \beta$$

for all $z \in E$. Using the fact that $\operatorname{Re} z \le |z|$ for all z, it follows that

$$\operatorname{Re}\left\{\frac{\sum_{k=1}^{\infty}(-1)^{k-1}k(k+2)^{-c}a_k z^{k+1}}{2\gamma(1-\alpha)-\sum_{k=1}^{\infty}(-1)^{k-1}(2\gamma-1)k(k+2)^{-c}a_k z^{k+1}}\right\} < \beta \qquad (2.4)$$

Now choose the values of z on the real axis so that $z^2(D_c f(z))'$ is real. Upon clearing the denominator in (2.3) and letting $z \to -1$ through real values, we obtain

$$\sum_{k=1}^{\infty} k(k+2)^{-c}a_k \le \beta\left\{2\gamma(1-\alpha) - \sum_{k=1}^{\infty}(2\gamma-1)\, k\,(k+2)^{-c}a_k\right\}$$

which is equivalent to

$$\sum_{k=1}^{\infty} k(k+2)^{-c}\left(1+2\beta\gamma-\beta\right)a_k. \le 2\beta\gamma(1-\alpha).$$

This completes the proof of Theorem 2.2

Corollary 2.1 Let the function *f(z)* defined by (1.14) be in the class $\sigma^*_{A,C}(\alpha,\beta,\gamma)$

$$a_k \le \frac{2\beta\gamma(1-\alpha)}{k(k+2)^{-c}(1+2\beta\gamma-\beta)} (k \ge 1)$$

Equality holds for the function of the form

$$f_k(z) = \frac{1}{z} + (-1)^{k-1} \frac{2\beta\gamma(1-\alpha)}{k(k+2)^{-c}(1+2\beta\gamma-\beta)} z^k$$

3. Distortion Theorems

Theorem 3.1. Let the function *f*(z) defined by (1.14) in the class $\sigma^*_{A,C}(\alpha,\beta,\gamma)$ then for 0 < |z| = r < 1,

$$\frac{1}{r} - \frac{2\beta r(1-\alpha)}{3^{-c}(1+2\beta r-\beta)} r \le |f(z)| \le \frac{1}{r} + \frac{2\beta r(1-\alpha)}{3^{-c}(1+2\beta r-\beta)} r$$

(3.1)

with equality for the function

$$f(z) = \frac{1}{z} + \frac{2\beta r(1-\alpha)}{3^{-c}(1+2\beta r-\beta)} z \quad \text{of } z = r, ir$$

(3.2)

Proof. suppose *f*(z) is in the class $\sigma^*_{A,C}(\alpha,\beta,\gamma)$. In view of

Theorem 2.1, we have

$$3^{-c}\left(1+2\beta\gamma-\beta\right)\sum_{k=1}^{\infty} a_k \le \sum_{k=1}^{\infty} k(k+2)^{-c}(1+2\beta\gamma-\beta)a_k \le 2\beta\gamma(1-\alpha)$$

which evidently yields

$$\sum_{k=1}^{\infty} a_k \le \frac{2\beta\gamma\left(1-\gamma\right)}{3^{-c}\left(1+2\beta\gamma-\beta\right)} \qquad (3.3)$$

Consequently we obtain

$$\left|f(z)\right| \le \frac{1}{r} + \sum_{k=1}^{\infty} a_k r^k \le \frac{1}{r} + r\sum_{k=1}^{\infty} a_k \le \frac{1}{r} + \frac{2\beta\gamma(1-\alpha)}{3^{-c}(1+2\beta\gamma-\beta)} r,$$

by (3.3) . This gives the right –hand inequality of (3.1)

Also.

$$\left|f(z)\right| \ge \frac{1}{r} - \sum_{k=1}^{\infty} a_k r^k \ge \frac{1}{r} - r\sum_{k=1}^{\infty} a_k \ge \frac{1}{r} - \frac{2\beta\gamma(1-\alpha)}{3^{-c}(1+2\beta\gamma-\beta)} r,$$

By (3.3), which gives the left hand of side of (3.1)

It can be easily seen that the function f(z) defined by (3.2) is external for the theorem.

Theorem 3.2: let the function f(z) defined by (1.14) be in the class $\sigma^{*}_{A,c}(\alpha,\beta,\gamma)$, then for $0<|z|=r<1$,

$$\frac{1}{r^2} - \frac{2\beta\gamma(1-\alpha)}{3^{-c}(1+2\beta\gamma-\beta)} \le \left|f^{1}(z)\right| \le \frac{1}{r^2} + \frac{2\beta\gamma(1-\alpha)}{3^{-c}(1+2\beta\gamma-\beta)} \qquad (3.4)$$

The result is sharp the external function being of the form (3.2)

Proof . From Theorem 2.1 we have

$$3^{-c}(1+2\beta\gamma-\beta)\sum_{k=1}^{\infty} k a_k \le \sum_{k=1}^{\infty} k(k+2)^{-c}(1+2\beta\gamma-\beta)a_k \le 2\beta\gamma(1-\gamma)$$

which evidently yields

$$\sum_{k=1}^{\infty} k a_k \le \frac{2\beta\gamma(1-\alpha)}{3^{-c}(1+2\beta r-\beta)} \qquad (3.5)$$

Consequently we obtain

$$\left|f^{1}(z)\right| \le \frac{1}{r^2} + \sum_{k=1}^{\infty} k a_k r^{k-1} \le \frac{1}{r^2} + \sum_{k=1}^{\infty} k a_k \le \frac{1}{r^2} + \frac{2\beta\gamma(1-\alpha)}{3^{-c}(1+2\beta r-\beta)}$$

Also

$$\left|f^{1}(z)\right| \geq \frac{1}{r^{2}} - \sum_{k=1}^{\infty} k a_{k} r^{k-1} \geq \frac{1}{r^{2}} - \sum_{k=1}^{\infty} k a_{k} \geq \frac{1}{r^{2}} - \frac{2\beta\gamma(1-\alpha)}{3^{-c}(1+2\beta r-\beta)}$$

This completes the proof of theorem 3.2

Corollary 3.1: Let the functions *f*(z) defined by (1.14) be in the class

$\sigma^{*}_{A,c}(\alpha,\beta,\gamma) = \sigma^{*}_{A,0}(\alpha,\beta,\gamma)$, then for $0<|z|=r<1$,

$$\frac{1}{r^{2}} - \frac{2\beta r(1-\alpha)}{(1+2\beta\gamma-\beta)} \leq \left|f^{1}(z)\right| \leq \frac{1}{r^{2}} + \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)}$$

The result is sharp.

Putting c=0 and $\beta=\gamma=1$ in theorem 2.1, we get

Corollary 3.2. Let the functions f(z) defined by (1.14) be in the class $\sigma^{*}_{A,C}(r\,1,1) = \sigma^{*}_{A,C}$,

then for 0 < |z| = r <1,

$$\frac{1}{r^{2}} - (1-\alpha) \leq |f'(z)| \leq \frac{1}{r^{2}} + (1-\alpha).$$

The result is sharp.

4.Class Preserving Integral operations

In this section we consider the class preserving integral operators of the form(1.15)

Theorem 4.1: let the function f(z) defined by (1.14) be in the class $\sigma^{*}_{A,C}(\alpha,\beta,\gamma)$, then

$$F(z) = sz^{-s-1}\int_{o}^{z} t^{s} f(t)\,dt = \frac{1}{z} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{s}{s+k+1} a_{k} z^{k}, s>o$$

belong to the class $\sigma^{*}_{A,C}(\lambda(\alpha,s),\beta,\gamma)$ where

$$\lambda(\alpha,s)=1-\frac{s(1-\alpha)}{(s+2)}$$

(4.1)

The result is sharp for

$$f(z)=\frac{1}{z}+\frac{2\beta\gamma(1-\alpha)}{3^{-c}(1+2\beta-\beta)}z$$

Proof : Suppose $f(z)\in\sigma^{*}_{A,C}(\alpha,\beta,\gamma)$, then

$$\sum_{k=1}^{\infty}k(k+2)^{-c}(1+2\beta\gamma-\beta)a_k\le 2\beta\gamma(1-\alpha)$$

In view of theorem 2.1, we shall find the largest values of λ for which

$$\sum_{k=1}^{\infty}\frac{k(k+2)^{-c}(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\lambda)}\quad\frac{s}{s+k+1}a_k\le 1$$

It suffices to find the range of λ for which

$$\frac{sk(k+2)^{-c}(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\lambda)(p+k+1)}\le\frac{k(k+2)^{-c}(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)}$$

Solving the above inequality for λ we obtain

$$\lambda\le 1-\frac{s(1-\alpha)}{(s+k+1)}$$

Since

$$\psi(k)=1-\frac{s(1-\lambda)}{(s+k+1)}$$

is an increasing functions of $k(k\ge 1)$,letting k=1 in (4.2) we obtain

$$\lambda=\Psi(1)=1-\frac{s(1-\alpha)}{(s+2)}$$

and the theorem follows at once.

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