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AN INITIALIZATION STRATEGY FOR IMPROVING NEWTON'S METHOD

Theodoula N. Grapsa

Abstract

Among the methods for solving a system of nonlinear equations, Newton's method is distinguished because of its significant advantage of converging quadratically. For this, a nonsingular Jacobian matrix and a good starting point are required, but they can rarely be available especially in application problems. On the other hand, Dimension Reducing method also of quadratic convergence works well even from initial points far away from the solution as well as in cases of singular or ill-conditioned Jacobian matrix. In this paper, DR and Newton methods are properly incorporated into a new algorithm of quadratic convergence to contribute to the important issue of initializing Newton's method. The quadratic convergence of the proposed method is proven and the numerical results on tested problems are promising.

Keywords:

Newton's method;
Dimension Reducing method;
Quadratic convergence;
Initialization;
Pivot points.

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1. Introduction

We consider the following system of nonlinear equations:

$$F(x) = 0 \qquad (1)$$

where $x=(x_1, x_2,\ldots, x_n)$ and $F = (f_1, f_2,\ldots, f_n): D\subset\Re^n \to \Re^n$ is continuously differentiable on an open neighborhood $D^* \subset D$ of a solution $x^* =(x_1^*, x_2^*,\ldots, x_n^*) \in D$ of the system (1).

The most well known method for solving systems of nonlinear equations is Newton's method, given by the relation:

$$x^{p+1} = x^p - F'(x^p)^{-1} F(x^p), \;\; p = 0,1,\ldots \qquad (2)$$

where $x^p = \left(x_1^p, x_2^p, \ldots, x_n^p\right)$ is an approximation of the solution x^* and $F'(x^p)$ is the Jacobian matrix of F at the point x^p. Newton's method for achieving its attractive advantage of quadratic convergence, appropriate initial points and nonsingular Jacobian matrix are required [5, 13, 16].

Several modified Newton's methods have been proposed either on achieving global convergence or facing the case of singular Jacobian matrix as well as for difficulties due to the computation of Jacobian matrix and a system of linear equations that must be solved at each iteration [1, 3, 4, 19]. Another way to improve Newton's performance is utilizing different local methods combinatorial. More details about such recent proposed combinations may be found in [15].

Another method for solving systems of nonlinear equations with promising results and also of quadratic convergence is Dimension Reducing (DR) method. In DR method appropriate points from the solution surfaces of function components, named pivot points [6, 7, 11] are exploited enforcing the method to converge even from points far away from the solution or in cases of singular or ill condition Jacobian matrix. More specifically, a point which is far from the solution through DR iteration may be moved quickly inside the region of convergence, contributing in the enlargement of the convergence area in comparison to Newton's method. The additional computational cost required for the extraction of the pivot points should be of no great concern since a low cost procedure can be employed and more importantly, because of DR method's potential for dramatically faster convergence from a "bad" initial point [2, 9].

However, for any iterative method the most important difficulty in practical problems is locating a good approximation of the solution. Since Newton's method for solving systems of nonlinear equations is a locally convergent method, finding a good initial point remains an important issue, mainly in the world of applications [10]. In this paper, an initialization strategy is introduced to contribute in this research area, whose an initial version was adressed in [8]. Specifically, a new method of quadratic convergence for solving systems of nonlinear equations is introduced, based on a combination of both Newton and DR methods taking into account each methods' characteristics. To this end, firstly from a random initial point, a few iterations of DR method take place, due to its promising behavior in the above-mentioned difficult cases for Newton's method, forcing the iteration to reach quickly to a point within a region near to a solution. Then, this point is used as a better starting point for Newton's iteration to continue until a termination criterion is reached. With DR iterations, forcing the first steps of the algorithm and at the same time giving the opportunity for the method to overcome points at which the Jacobiam matrix is singular or ill conditioned, Newton's method is shown improved.

The structure of this paper is the following: In Section 2 we represent a briefly presentation of DR method. In Section 3 we present and formulate the proposed method. Moreover, its convergence theorem is proven. In Section 4 numerical results are given and in Section 5 some conclusions and ideas for future work are presented.

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2. The DR Method

Next, DR iterative form will be given as mentioned in [9].

Let $y^p = \left(x_1^p, x_2^p, \ldots, x_{n-1}^p\right) = \left[x_i^p\right]$, $i = 1,2,\ldots,n-1$ in p iteration. The pivot points $x_{pivot}^{p,i} \equiv \left(y^p; x_n^{p,i}\right) = \left(x_1^p, x_2^p, \ldots, x_{n-1}^p, x_n^{p,i}\right)$ of function components $f_i(x)$, $i = 0,1,\ldots,n$ are points on the solution surfaces of $f_i(x)$ and thus $f_i\left(x_{pivot}^{p,i}\right) = 0$. The components $x_n^{p,i}$, $i = 1,2,...,n$ of the pivot points are extracted by solving the one-dimension equations:

$$f_i\left(x_1^p, x_2^p, \ldots, x_{n-1}^p, \cdot\right) = 0 \qquad i = 0,1,\ldots,n \tag{3}$$

in which the components $x_1^p, x_2^p, \ldots, x_{n-1}^p$ are fixed. Of course, any other set of n-1 components x_i may be choosen to be fixed for producing the unkown component of pivot points, [7].

Taking $x^0 = \left(x_1^0, x_2^0, \ldots, x_n^0\right)$ as the initial guess of the solution x^*, according to the DR iteration for the computation of n-1 components of x^* we have:

$$y^{p+1} = y^p + A_p^{-1} V_p, \quad p = 0,1,2,... \tag{4}$$

where

$$V_p = [v_i] = \left[x_n^{p,i} - x_n^{p,n}\right], \quad i = 1,\ldots,n-1$$

and

$$A_p = \left[a_{ij}\right] = [\partial_j v_i\left(y^p; x_n^{p,i}\right) - \partial_j v_n\left(y^p; x_n^{p,n}\right)] =$$

$$= [\frac{\partial_j f_i\left(y^p; x_n^{p,i}\right)}{\partial_n f_i\left(y^p; x_n^{p,i}\right)} - \frac{\partial_j f_n\left(y^p; x_n^{p,n}\right)}{\partial_n f_n\left(y^p; x_n^{p,n}\right)}], \qquad i, j = 1,2,...,n-1$$

Then, after a desired number of $p = m$ iterations of (4), the $n-th$ component of x^* is approximated by

$$x_n^{m+1} = x_n^{m,n} - \sum_{j=1}^{n-1} \left(x_j^{m+1} - x_j^m\right) \partial_j f_n\left(y^m; x_n^{m,n}\right) / \partial_n f_n\left(y^m; x_n^{m,n}\right). \tag{5}$$

Remark 1: Any method may be utilized to solve the one-dimensional equations (3). In this work for the computation of the n-th components $x_n^{p,i}$, $i = 1,2,...,n$ of the pivot points we use a bisection method, which is based on the algebraic signs of function values [7, 9, 17, 18].

3. The Proposed Algorithm and its Convegence

As we have already mentioned the proposed method, called Combined Newton-DR (CoNDR) method, is a combination of two quadratically convergent methods for a better performance of Newton's method. This is achieved by feeding Newton's method with a better initial point extracted from DR method which is used as a preprocessing step. The idea is implemented as per the following algorithm.

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Algorithm 1: The Proposed Algorithm (CoNDR)

step 1: F(x): the vector of functions components $f_1(x), f_2(x),\dots, f_n(x)$, F′(x): the Jacobian matrix of F(x), x^0 the initial point, maxiterDR: the maximum number of DR iterations, maxiterN: the maximum number of Newton's iterations, root: the solution point, tolx and tolf: the accuracy.

step 2: k=0

step 3: while $k < \max iterDR$ do

call DR algorithm $(F(x), F'(x), x^k, tolx, tolf, x^{k+1})$

if ($\left\|x^{k+1} - x^k\right\| \le tolx \wedge \left\|F\left(x^{k+1}\right)\right\| \le tolf$) then

go to step 7

else

$k = k+1$

endif

endwhile

step 4: p=0

step 5: $x^0 = x^{\max iterDR}$

step 6: while $p < \max iterN$ do

call Newton algorithm $(F(x), F'(x), x^p, tolx, tolf, x^{p+1})$

if (($\left\|x^{p+1} - x^p\right\| \le tolx \wedge \left\|F\left(x^{p+1}\right)\right\| \le tolf$)) then

go to step 7

else

$p = p+1$

endif

endwhile

step 7: return

Notice that if maxiterDR=0 the algorithm coincides with Newton algorithm and when the algorithm terminates in an iteration $k < \max iterDR$ then it coincides with DR algorithm.

The Convergence Theorem of CoNDR method

Since the DR iterations consist the preprocessing step to feed the Newton's method with a different initial point, the convergence theorem is that of Newton's method.

Theorem

Let $F = \left(f_1, f_2,\dots, f_n\right): D \subset \Re^n \to \Re^n$ is twice continuously differentiable on an open neighborhood $D^* \subset D$ of a solution $x^* = \left(x_1^*, x_2^*,\dots, x_n^*\right) \in D$. Then the iterations x^p, $p = 0,1,\dots$ of the method given by (2) will converge to x^* provided the Jacobian matrix of F at x^* is nonsingular and also provided the initial guess x^0 is sufficiently close to x^*. Moreover the order of convergence will be two.

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Proof
See [12, 14, 16].

4. Numerical Results

The proposed CoNDR algorithm has been implemented using Fortran programs on a PC Intel Core i5. Our numerical experience has shown that the CoNDR method provides promising results, improving Newton's method by reducing both the number of iterations and the computational cost. In some cases the reduction in iterations and computational cost may be much more than 50% even with only one iteratation of DR method.

Table 1, Table 2 and Table 3 present correspondingly the numerical results for three examples with accuracy equal to $\varepsilon = 10^{-14}$ as referred to [9], with characteristics as follows:

Example 1: The Jacobian matrix is nonsingular, and the difficulty is that at some points the function values cannot be achieved accurately.
Example 2: The Jacobian matrix is singular.
Example 3: The Jacobian matrix is nonsingular and ill-conditioned.

Example 1

$$f_1(x_1, x_2, x_3) = x_1^3 - x_1 x_2 x_3 = 0$$
$$f_2(x_1, x_2, x_3) = x_2^2 - x_1 x_3 = 0$$
$$f_3(x_1, x_2, x_3) = 10 x_1 x_3 + x_2 - x_1 - 0.1 = 0$$

Example 2

$$f_1(x_1, x_2, x_3) = x_1 x_3 - x_3 e^{x_1^2} + 10^{-4} = 0$$
$$f_2(x_1, x_2, x_3) = x_1 \left(x_1^2 + x_2^2 \right) + x_2^2 \left(x_3 - x_2 \right) = 0$$
$$f_3(x_1, x_2, x_3) = x_1^3 + x_3^3 = 0$$

Example 3

$$f_1(x_1, x_2, x_3, x_4, x_5) = 2x_1 + x_2 + x_3 + x_4 + x_5 - 6 = 0$$
$$f_2(x_1, x_2, x_3, x_4, x_5) = x_1 + 2x_2 + x_3 + x_4 + x_5 - 6 = 0$$
$$f_3(x_1, x_2, x_3, x_4, x_5) = x_1 + x_2 + 2x_3 + x_4 + x_5 - 6 = 0$$
$$f_4(x_1, x_2, x_3, x_4, x_5) = x_1 + x_2 + x_3 + 2x_4 + x_5 - 6 = 0$$
$$f_5(x_1, x_2, x_3, x_4, x_5) = x_1 x_2 x_3 x_4 x_5 - 1 = 0$$

As a notation to the numerical tables we introduce:
" x^0 " the initial point,
"IT" the total number of iterations required in each case,
"FE" the total number of function evaluations (including derivatives),
"AS" the total number of algebraic signs for the computation of n-th component $x_n^{p,i}$ of pivot points,
" r_i " the root to which each method converges.

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In all Tables the CoNDR method has been implemented with maximum number of DR iterations (maxiterDR) to be 1 and 2.

Table 1: Example 1: Numerical Results for Newton and CoNDR methods with maxiterDR=1 and maxiterDR=2

		Newton			CoNDR			
maxiterDR	x^0	IT	FE	r_i	IT	FE	AS	r_i
1	(0.4,0.5,0.5)	53	636	r_2	7	93	30	r_1
	(-4,-2,1)	33	396	r_2	6	81	30	r_1
	(-1,-2,0.6)	51	612	r_1	6	81	30	r_2
	(2,-2,-2)	43	516	r_1	14	177	30	r_2
2	(0.4,0.5,0.5)	53	636	r_2	7	102	60	r_1
	(-4,-2,1)	33	396	r_2	6	90	60	r_1
	(-1,-2,0.6)	51	612	r_1	5	78	60	r_2
	(2,-2,-2)	43	516	r_1	6	90	60	r_2

Table 2: Example 2: Numerical Results for Newton and CoNDR methods with maxiterDR=1 and maxiterDR=2

		Newton			CoNDR			
maxiterDR.	x^0	IT	FE	r_i	IT	FE	AS	r_i
1	(2,2,2)	42	504	r	39	477	60	r
	(3,3,3)	122	1464	r	40	489	60	r
	(3,3,5)	92	1104	r	41	501	60	r
	(4,4,4)	73	876	r	47	573	60	r
2	(2,2,2)	42	504	r	39	477	60	r
	(3,3,3)	122	1464	r	40	489	60	r
	(3,3,5)	92	1104	r	41	501	60	r
	(4,4,4)	73	876	r	47	573	60	r

Table 3: Example 3: Numerical Results for Newton and CoNDR methods with maxiterDR=1 and maxiterDR=2

		Newton			CoNDR			
maxiterDR	x^0	IT	FE	r_i	IT	FE	AS	r_i
1	(-8,-3,4,2,1.5)	85	2550	r_3	8	265	50	r_1
	(10,3,4,2,1.5)	83	2490	r_3	8	265	50	r_1
	(-0.2,-0.2,-0.2,-0.2,-0.2)	36	1080	r_3	13	415	50	r_3
	(-0.1,-0.1,-0.1,-0.1,-0.1)	49	1470	r_3	21	655	50	r_3
2	(-8,-3,4,2,1.5)	85	2550	r_3	8	290	100	r_1
	(10,3,4,2,1.5)	83	2490	r_3	8	290	100	r_1
	(-0.2,-0.2,-0.2,-0.2,-0.2)	36	1080	r_3	11	380	100	r_3
	(-0.1,-0.1,-0.1,-0.1,-0.1)	49	1470	r_3	19	620	100	r_3

5. Conclusion

The proposed method is a combination of two quadratically convergent methods for a better performance of Newton's method. This is achieved by feeding Newton's method with a better initial point which is extracted from DR method used as preprocessing step. Although the idea seems to be simple, the numerical results are significant as DR and Newton's iterations are efficiently balanced. Bypassing a bad initial point, the proposed method enforces a fast convergence of Newton's method and furthermore contributes even in cases where a singular or an ill-conditioned Jacobian matrix exists. An issue that must be further investigated is to find an estimation for the number of DR iterations to be implemented in order to give the best starting point to Newton's method and result to its acceleration as well as to reduce as much as possible its computational cost. Also, the incorporation of DR method in other local methods may be further investigated since the initialization of a local method is still an active research field.

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