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TWO UNIQUE PRIMITIVE PYTHAGOREAN TRIPLES FROM EVERY INTEGER

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ABSTRACT

In this paper we show that for every integer, there are two unique primitive solutions to the classical Pythagorean equation. These solutions have two interesting points. The two unique primitive triples correspond to ***every*** *integer, with no additional conditions. Two, for each of the solutions, C-B is the set of all odd integers squared.*

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1. PYTHAGOREAN TRIPLES

Consider the Pythagorean triple:

$$A^2 + B^2 = C^2$$

where A, B, and C are all positive integers. The triple is a primitive triple if A, B and C are relatively prime[a].

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The general Euclidean solution is, $A = M^2 - N^2$, $B = 2MN$, $C = M^2 + N^2$, where $M > N > 0$. To generate primitive triples, one must add two conditions, M and N must be relatively prime, and either M or N must be even[b].

In Appendix A we show that for every integer, I, there are two unique primitive solutions (each of which is just a special case of the general Euclidean solution[c]). There are three interesting points about these solutions. One, they generate two unique primitive triples for **every** integer, with no additional conditions. Two, for each of the solutions, C - B is the set of all odd integers squared. Three, the solutions uses an analysis of C – A, rather than the factoring approach used in the standard Euclidean analysis.

One solution is: $A = 4I^2 – 1$, $B = 4I$, $C = 4I^2 + 1$. An interesting result, is that $C - B = (2I - 1)^2$. In other words, for the set of solutions, C-B is the set of all odd integers squared. Another interesting result, is that $C/B = I + 1/4I$, which approaches I as I gets larger.

A second solution, totally different than the first solution[d], is: $A = (2I + 1)^2 – 4$, $B = 4(2I + 1)$, $C = (2I +1)^2 + 4$. An interesting result, is that again $C - B = (2I - 1)^2$, so that C-B is the set of all odd integers squared. Another interesting result, is that $2(C/B -1) = I + 2/ (2I + 1)$, which approaches I as I gets larger.

Table 1 presents a summary. Table 2 provides the mapping of the two solutions for every integer from 1 to 50.

Table 1. Summary of Solutions

	Solution 1	Solution 2
A(I)	$A = 4I^2 – 1$	$A = (2I + 1)^2 – 4$
B(I)	$B = 4I$	$B = 4(2I + 1)$
C(I)	$C = 4I^2 + 1$	$C = (2I +1)^2 + 4$
Euclidean Solution	N=1, M=2I	N=2, M=2I+1
C- A	2	8
$(C-B)^{1/2}$	2I – 1 (the set of all odd positive integers)	
as $I \to \infty$	$C/B \to I$	$2(C/B – 1) \to I$

Table 2. Mapping of Two Solutions for Every Integer From 1 to 50

	Case 1						Case 2					
I	A	B	C	C-A	SQRT(C-B)	C/B	A	B	C	C-A	SQRT(C-B)	2*(C/B -1)
1	3	4	5	2	1	1.25	5	12	13	8	1	1.67
2	15	8	17	2	3	2.13	21	20	29	8	3	2.40
3	35	12	37	2	5	3.08	45	28	53	8	5	3.29
4	63	16	65	2	7	4.06	77	36	85	8	7	4.22
5	99	20	101	2	9	5.05	117	44	125	8	9	5.18
6	143	24	145	2	11	6.04	165	52	173	8	11	6.15
7	195	28	197	2	13	7.04	221	60	229	8	13	7.13
8	255	32	257	2	15	8.03	285	68	293	8	15	8.12
9	323	36	325	2	17	9.03	357	76	365	8	17	9.11
10	399	40	401	2	19	10.03	437	84	445	8	19	10.10
11	483	44	485	2	21	11.02	525	92	533	8	21	11.09
12	575	48	577	2	23	12.02	621	100	629	8	23	12.08
13	675	52	677	2	25	13.02	725	108	733	8	25	13.07
14	783	56	785	2	27	14.02	837	116	845	8	27	14.07
15	899	60	901	2	29	15.02	957	124	965	8	29	15.06
16	1,023	64	1,025	2	31	16.02	1,085	132	1,093	8	31	16.06
17	1,155	68	1,157	2	33	17.01	1,221	140	1,229	8	33	17.06
18	1,295	72	1,297	2	35	18.01	1,365	148	1,373	8	35	18.05
19	1,443	76	1,445	2	37	19.01	1,517	156	1,525	8	37	19.05
20	1,599	80	1,601	2	39	20.01	1,677	164	1,685	8	39	20.05
21	1,763	84	1,765	2	41	21.01	1,845	172	1,853	8	41	21.05
22	1,935	88	1,937	2	43	22.01	2,021	180	2,029	8	43	22.04
23	2,115	92	2,117	2	45	23.01	2,205	188	2,213	8	45	23.04
24	2,303	96	2,305	2	47	24.01	2,397	196	2,405	8	47	24.04
25	2,499	100	2,501	2	49	25.01	2,597	204	2,605	8	49	25.04
26	2,703	104	2,705	2	51	26.01	2,805	212	2,813	8	51	26.04
27	2,915	108	2,917	2	53	27.01	3,021	220	3,029	8	53	27.04
28	3,135	112	3,137	2	55	28.01	3,245	228	3,253	8	55	28.04
29	3,363	116	3,365	2	57	29.01	3,477	236	3,485	8	57	29.03
30	3,599	120	3,601	2	59	30.01	3,717	244	3,725	8	59	30.03
31	3,843	124	3,845	2	61	31.01	3,965	252	3,973	8	61	31.03
32	4,095	128	4,097	2	63	32.01	4,221	260	4,229	8	63	32.03
33	4,355	132	4,357	2	65	33.01	4,485	268	4,493	8	65	33.03
34	4,623	136	4,625	2	67	34.01	4,757	276	4,765	8	67	34.03
35	4,899	140	4,901	2	69	35.01	5,037	284	5,045	8	69	35.03
36	5,183	144	5,185	2	71	36.01	5,325	292	5,333	8	71	36.03
37	5,475	148	5,477	2	73	37.01	5,621	300	5,629	8	73	37.03
38	5,775	152	5,777	2	75	38.01	5,925	308	5,933	8	75	38.03
39	6,083	156	6,085	2	77	39.01	6,237	316	6,245	8	77	39.03
40	6,399	160	6,401	2	79	40.01	6,557	324	6,565	8	79	40.02
41	6,723	164	6,725	2	81	41.01	6,885	332	6,893	8	81	41.02
42	7,055	168	7,057	2	83	42.01	7,221	340	7,229	8	83	42.02
43	7,395	172	7,397	2	85	43.01	7,565	348	7,573	8	85	43.02
44	7,743	176	7,745	2	87	44.01	7,917	356	7,925	8	87	44.02
45	8,099	180	8,101	2	89	45.01	8,277	364	8,285	8	89	45.02
46	8,463	184	8,465	2	91	46.01	8,645	372	8,653	8	91	46.02
47	8,835	188	8,837	2	93	47.01	9,021	380	9,029	8	93	47.02
48	9,215	192	9,217	2	95	48.01	9,405	388	9,413	8	95	48.02
49	9,603	196	9,605	2	97	49.01	9,797	396	9,805	8	97	49.02
50	9,999	200	10,001	2	99	50.01	10,197	404	10,205	8	99	50.02

Appendix:

Lemma 1: For three integers A + B = C, all three must be even, or two must be odd and one must be even.

It is impossible for all three to be odd, or for two to be even and one to be odd.

Proof: Assume all three are odd, A= 2S + 1, B = 2T + 1, C = 2U + 1. Then A = B + C becomes:

$$2(S + T + 1 - U) = 1$$

Since the left-hand side of the equation is even, the right side can't equal 1, which is odd. So, all three terms can't be odd.

Assume two terms are even and one is odd, A= 2S, B = 2T, C = 2U + 1. Then A = B + C becomes:

$$2(S + T - U) = 1$$

Since the left-hand side of the equation is even, the right side can't equal 1, which is odd. So it is impossible for two terms to be even and one odd.

APPENDIX A: TWO UNIQUE PRIMITIVE PYTHAGOREAN TRIPLES SOLUTIONS

Since A, B, and C are relatively prime, and any even number squared is even, it follows that A, B and C cannot all be even. Since any odd number squared, is odd, from Lemma 1, it follows that of the three terms, A, B and C, two are odd and one is even.

Furthermore, C cannot be even (see Appendix B). Therefore, if a Pythagorean triple is primitive, then C must be odd, and of the terms A and B, one is odd and one is even.

Without loss of generality, let A be odd, and let B be even, B = 2N. In Appendix C we prove that N must be even, so let N = 2L and, therefore, **B = 4L**.

Since C > A, and both A and C are odd, we define C = A + 2G, and the Pythagorean triple becomes Equation 1:

$$4L^2/G = A + G$$

Since A and G are integers, $4L^2/G$ must be an integer. There are only 4 possible ways $4L^2/G$ can be an integer[e].

Case 1: G = 1.

Case 2: G = 4.

Case 3: G = 2.

Case 4: G and L have a common factor.

For Case 1, Equation1 becomes $\mathbf{A = 4L^2 - 1}$. Since C = A +2G, we get that C $\mathbf{= 4L^2 + 1.}$

Since L is any integer, we substitute I for L.

For Case 2, Equation1 becomes $L^2 = A + 4$. By construction Ais odd, thus L^2 must be odd (Lemma 1) so define L = 2P + 1. We then get A **= (2P + 1)^2 – 4.**Since C = A +2G, we get that **C = (2P +1)^2 + 4.** Since, B = 4L, we get **B = 4(2P + 1).** Since P is any integer, we substitute I for P.

APPENDIX B: C CANNOT BE EVEN

If C is even, define C = 2Y. Since both A and B are odd, define A = B + 2X, where $X \geq 0$. The Pythagorean triple then becomes

$$B^2/2 + BX + X^2 = Y^2$$

Since all the terms besides $B^2/2$ are integers, $B^2/2$ must be in integer. So, B must be even. But this contradicts our assumption that B is odd. So, it is impossible for C to be even.

APPENDIX C: PROOF THAT WHEN B = 2N, N MUST BE EVEN

Assume N is odd, so N = 2G + 1. Since C > A, let C = A + X. The Pythagorean triple then becomes

$$2(2G + 1)^2 = AX + X^2/2$$

Since all the terms other than $X^2/2$ are integers, $X^2/2$must be in integer. Therefore, X must be even. Define X = 2P.

Substituting 2P for X and simplifying yields

$$(2G + 1)^2 = AP + P^2$$

Since $(2G + 1)^2$ is odd, From Lemma 1 of the two terms AP or P^2, one must be odd and one must be even.

If P is even, then both AP and P^2 are even. So, P cannot be even. If P is odd, P^2 is odd and since by construction A is odd, AP is also odd, so P cannot be odd. Since it is impossible for P to not be even and not bze odd, it follows that N cannot be odd.

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[a]In other words, there is no integer Z, Z > 1, for which A/Z, B/Z and C/Z are all integers. Alternatively, this can be formulated as GCD (A, B, C) =1 (where GCD is the greatest common denominator). This will only be true only if GCD (A, B) = 1, GCD (A, C) = 1 and GCD (B, C) = 1. Since if any two of the terms have a common factor, Z >1, for instance, A=ZD and B=ZE. Then we get$D^2 + E^2 = (C/Z)^2$. Since D and E are both integers, C/Z must be an integer, so Z is also a factor of C.

[b]For an alternative formulation, see Douglas W. Mitchell. "85.27 An Alternative Characterisation of All Primitive Pythagorean Triples." *The Mathematical Gazette*, vol. 85, no. 503, 2001, pp. 273–275. *JSTOR*, JSTOR, www.jstor.org/stable/3622017.

[c]For solution 1: N =1 and M = 2I. For solution 2: N =2 and M = 2I + 1.

[d]All the triples in Solution 2 are different than the triples in Solution 1, since in Solution 1, C = A + 2, while in Solution 2, C = A + 8.

[e]Case 3 is an impossible solution, since for G=2, Equation1 becomes $2L^2 = A + 2$. Since $2L^2$ and 2 are both even, A must be even (Lemma 1). But by construction, A is odd. So, Case 2 is impossible. We do not analyze Case 4.