International Journal of Engineering, Science and Mathematics
Vol. 6 Issue 5, September 2017,
ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijesm.co.in, Email: ijesmj@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gage as well as in Cabell's Directories of Publishing Opportunities, U.S.A

USE OF GENERALIZED ERDELYI- KOBER OPERATORS IN WEBER-ORR TRANSFORMS

Dr. Poonam Kumari
Associate Professor, Dept. of Mathematics
University College, Kurukshetra University, Kurukshetra

ABSTRACT

In this paper we have discussed formal integral transform which is generalization of Weber-Orr transforms and its inverse by making use of generalized Erdelyi-Kober operators.

Keywords :Weber-Orr transform, Erdelyi-Kober Operators.

1. INTRODUCTION

We have established generalization of Weber-Orr transform and its inverse transform due to Nasim [5] by making use of generalized Erdelyi-Kober operators defined earlier by Lowndes [4]. Now, we introduce generalization of Weber-Orr transform $W_{\mu,\upsilon}^{k}$ [;] of arbitrary order $(\mu,\ \upsilon)$ and its inverse $W_{\mu,\upsilon}^{k^{-1}}$ [;] by the following expressions

$$\hat{f}(t)=W_{\mu,\upsilon}^{k}[f(s);t]=\int_{a}^{\infty} R_{\mu,\upsilon}^{k}(t;\ s,\ a)\ s\ f(s)ds \qquad \ldots(1.1)$$

and

$$f(s)=W_{\mu,\upsilon}^{k^{-1}}\left[\hat{f}(t);\ a\right]=\int_{0}^{\infty} \frac{R_{\mu,\upsilon}^{k}(t;\ s,\ a)}{J_{\upsilon}^{2}(ta)+Y_{\upsilon}^{2}(ta)}\ t\ \hat{f}(t)dt \qquad \ldots(1.2)$$

where,

$$R_{\mu,\upsilon}^{k}(t;\ s,\ s)=\left[J_{\mu}(ts)Y_{\upsilon}(ta)-J_{\upsilon}(ta)Y_{\mu}(ts)\right]{}_0F_1\left[\upsilon-\mu;\ \frac{k^2(x^2-s^2)}{4}\right].$$

Also, f is continuous on $(a,\ \infty)$ and $\left|\int_a^{\infty} x^{\frac{1}{2}} f(x)dx\right| < \infty$.

We deal with generalized Weber-Orr transform $W_{\mu,\nu}^{k}[;]$ for the parameter μ. Many known results have been given as particular cases. The results obtained are mostly used in Mathematical Physics in general.

2. THE TRANSFORM $W_{\upsilon-\alpha,\upsilon}^{k}[;],\ \alpha>0$

Lemma 2.1. If $0<\alpha<\frac{1}{2}\upsilon+\frac{3}{4}$, then

$$R_{\upsilon-\alpha,\upsilon}^{k}(t;\ s,\ a)=a^{\upsilon-\alpha}t^{\alpha}k^{1-\alpha}\int_s^{\infty} x^{1-\upsilon}(x^2-t^2)^{\frac{\alpha-1}{2}}.\ R_{\upsilon,\upsilon}^{k}(t;\ x,\ a)\ J_{\alpha-1}\left[k(x^2-t^2)^{\frac{1}{2}}\right]dx \qquad \ldots(2.1)$$

where

$$R_{\mu,\upsilon}^{k}(t;\ s,\ a)=\left[J_{\mu}(ts)Y_{\upsilon}(ta)-J_{\upsilon}(ta)Y_{\mu}(ts)\right]{}_0F_1\left[\upsilon-\mu;\ k\left(\frac{x^2-s^2}{4}\right)\right].$$

By making use of standard result Erdely et.al. [1, p. 25, 104] the result can be obtained.

Lemma 2.2. If $0<\alpha<\frac{1}{2}\upsilon+\frac{3}{4}$, $\int_a^{\infty}\left|x^{\alpha+\frac{1}{2}} f(x)\right|dx<\infty$

and $F(x) = x^{\alpha-\upsilon} k^{1-\alpha} \int_a^x s^{1+\upsilon-\alpha} (x^2 - s^2)^{\frac{\alpha-1}{2}} \,.\, J_{\alpha-1}\left[k(x^2-s^2)^{\frac{1}{2}}\right] f(s) ds$ ...(2.2)

then,

$$\int_a^\infty x^{\frac{1}{2}-\alpha} F(x) dx = \frac{2^{-\alpha}}{\Gamma(\alpha)} B\left(\frac{1}{2}\upsilon + \frac{1}{4} - \alpha, \alpha\right) \int_0^\infty s^{\frac{1}{2}+\alpha}$$

$$\cdot\ {}_0F_1\left[\alpha; \frac{k^2(x^2-s^2)}{4}\right] f(s) ds \qquad \text{...(2.3)}$$

where,

$$B\left(\frac{1}{2}\upsilon + \frac{1}{4} - \alpha, \alpha\right) = \frac{\Gamma\left(\frac{1}{2}\upsilon + \frac{1}{4} - \alpha\right)\Gamma(\alpha)}{\Gamma\left(\frac{1}{2}\upsilon + \frac{1}{4}\right)}.$$

Proof: From (2.2), we have,

$$\int_a^\infty x^{\frac{1}{2}-\alpha} F(x) dx = k^{1-\alpha} \int_a^\infty x^{\frac{1}{2}-\upsilon} dx \int_a^x s^{1+\upsilon-\alpha} (x^2-s^2)^{\frac{\alpha-1}{2}}$$

$$.\ J_{a-1}\left[k(x^2-s^2)^{\frac{1}{2}}\right] f(s) ds$$

On changing the order of integration and evaluating integral, we obtain

$$\int_a^\infty x^{\frac{1}{2}-\alpha} F(x) dx = \frac{2^{-\alpha}}{\Gamma(\alpha)} B\left(\frac{1}{2}\upsilon + \frac{1}{4} - \alpha, \alpha\right) \int_a^\infty s^{\frac{1}{2}+\alpha}$$

$$\cdot\ {}_0F_1\left[\alpha; \frac{k^2(x^2-s^2)}{4}\right] f(s) ds$$

The result is valid for the extended range, $0 < \alpha < \frac{1}{2}\upsilon + \frac{3}{4}$ by analytic continuation as a $\alpha \neq \frac{1}{2}\upsilon + \frac{1}{4}$ and $\upsilon > -\frac{1}{2}$.

Corollary: $\left|\int_a^\infty x^{\frac{1}{2}-\alpha} F(x) dx\right| \le K \int_a^\infty \left|s^{\frac{1}{2}+\alpha} f(s)\right| ds < \infty$ ...(2.4)

Where, K denotes a constant.

Theorem 2.1. If $0 < \alpha < \frac{1}{2}\upsilon + \frac{3}{4},\ \upsilon > \frac{1}{2}, \int_a^\infty \left|x^{\frac{1}{2}+\alpha} f(x)\right| dx < \infty$

and $\hat{f}(t) = W^k_{\upsilon-\alpha,\upsilon}[f(s); t]$, then $x^{-\alpha} F(x) = W^{k-1}_{\upsilon-\alpha,\upsilon}\left[t^{-\alpha} \hat{f}(t);\ x\right]$

where

$$x^{-\alpha} F(x) = k^{1-\alpha} x^{-\upsilon} \int_a^x s^{1+\upsilon-\alpha} (x^2-s^2)^{\frac{\alpha-1}{2}}$$

$$.\ J_{\alpha-1}\left[k(x^2-s^2)^{\frac{1}{2}}\right] f(s) ds$$

Proof: From (1.1), we have,

International Journal of Engineering, Science and Mathematics
Vol. 6 Issue 5, September 2017,
ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijesm.co.in, Email: ijesmj@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gage as well as in Cabell's Directories of Publishing Opportunities, U.S.A

$$f(\hat{t})=\int_a^\infty R^k_{\upsilon-\alpha,\upsilon}(t;\, s,\, a)sf(s)ds$$

Here, the integral exists and is absolutely convergent due to the condition stated with the theorem, on making use of Lemma (2.1), we have,

$$\hat{f}(t)=t^\alpha k^{1-\alpha}\int_a^\infty s^{1+\upsilon-\alpha}f(s)ds\int_a^\infty x^{1-\upsilon}\left(x^2-s^2\right)^{\frac{\alpha-1}{2}}$$

$$.\, R^k_{\upsilon,\upsilon}(t;\, x,\, a)\, J_{\alpha-1}\left[k\left(x^2-s^2\right)^{\frac{1}{2}}\right]dx$$

By changing the order of integration, valid due to absolute convergence, and using (2.2), we have,

$$t^{-\alpha}\hat{f}(t)=\int_a^\infty x^{1-\alpha}R^k_{\upsilon,\upsilon}(t;\, x,\, a)\, F(x)dx$$

$$=W^k_{\upsilon,\upsilon}\left[x^{-\alpha}F(x);\, t\right]$$

On applying inversion formula (1.2), valid because of (2.4), we obtain the required result

$$x^{-\alpha}F(x)=W^{k^{-1}}_{\upsilon,\upsilon}\left[t^{-\alpha}\hat{f}(t);\; x\right]$$

Before going to next Lemma, we shall first define generalized Erdelyi-Kober operators due to Lowndes [4] as

$$ {}_a^xA_k(\eta,\, \alpha)\, f(u)=2^\alpha x^{-2\alpha}x^{-2\eta-2\alpha}k^{1-\alpha}\int_a^x u^{2\eta+1}\left(x^2-u^2\right)^{\frac{\alpha-1}{2}}$$

$$.\, J_{\alpha-1}\left[k\left(x^2-u^2\right)^{\frac{1}{2}}\right]f(u)du,\;\; \alpha>0 \qquad \ldots(2.5)$$

and

$$ {}_a^xA_k(\eta,\, \alpha)\, f(u)=x^{-1-2\eta-2\alpha}D_x^m\left[x^{2m+2\alpha+2\eta+1}\, {}_a^xA_k(\eta,\, \alpha+m)\; f(u)\right] \qquad \ldots(2.6)$$

for $\alpha<0,\;\; 0\le\alpha+m<1,\; D_x\equiv\frac{1}{2}\frac{d}{dx}\frac{1}{x}$ and $m=1,\, 2,\, \ldots.$

If ${}_a^xA_k^{-1}(\eta,\, 0)$ is the identity operator then the inverse operator is defined as

$$ {}_a^xA_k^{-1}(\eta,\, \alpha)=\, {}_a^xA_k(\eta+\alpha,\; -\alpha) \qquad \ldots(2.7)$$

Lemma 2.3. If $F(x)=k^{1-\alpha}\, x^{\alpha-\upsilon}\int_a^x s^{1+\upsilon-\alpha}\left(x^2-s^2\right)^{\frac{\alpha-1}{2}}J_{\alpha-1}\left[k\left(x^2-s^2\right)^{\frac{1}{2}}\right]$

$$.\;\; f(s)ds$$

then,

$$f(x)=\frac{2^{1+\alpha}}{\Gamma(m-\alpha)}x^{\alpha-\upsilon-1}D_x^m\left[x\int_a^x u^{\upsilon-\alpha+1}\left(x^2-u^2\right)^{m-\alpha-1}\right.$$

$$\left. .\;\; {}_0F_1\left(m-\alpha;\, \frac{k^2\left(x^2-u^2\right)}{4}\right)F(u)du\right]$$

where $\alpha>0,\;\; 0\le m-\alpha<1,\;\;\; m=1,\, 2,\, \ldots$

Proof: Making use of (2.5), we have

$$F(x)=\left(\frac{x^2}{2}\right)^\alpha {}_a^xA_k\left(\frac{\upsilon-\alpha}{2},\;\alpha\right)\, f(u),\;\alpha>0$$

From (2.6), (2.7), we have,

$$f(x)={}_a^x A_k^{-1}\left(\frac{\upsilon-\alpha}{2},\ \alpha\right)\left[\left(\frac{2}{u^2}\right)^{\alpha}F(u)\right]$$

$$={}_a^x A_k\left(\frac{\upsilon+\alpha}{2},\ -\alpha\right)\left[\left(\frac{2}{u^2}\right)^{\alpha}F(u)\right] \qquad \ldots(2.8)$$

$$=\frac{2^{1+\alpha}}{\Gamma(m-\alpha)}x^{\alpha-\upsilon-1}D_x^m\left[x\int_a^x u^{\upsilon-\alpha+1}\left(x^2-u^2\right)^{m-\alpha-1}\right.$$

$$\left.\cdot\ {}_0F_1\left(m-\alpha;\ \frac{-k^2\left(x^2-u^2\right)}{4}\right)F(u)du\right]$$

If we set $\alpha=m,\ m=1,\ 2,\ \ldots$ as a particular case and ${}_a^x A_k(\eta,\ 0)$ using the identify operator, from (2.8) we obtain,

$$f(x)=x^{m-\upsilon}\left(\frac{1}{x}\frac{d}{dx}\right)^k\left[x^{\upsilon-m}F(x)\right]$$

Lemma 2.4.

$$x^{\upsilon-m}f(x)=\left(\frac{1}{x}\frac{d}{dx}\right)^m\left[x^{\upsilon-m}F(x)\right] \qquad \ldots(2.9)$$

where *f* and *F* are defined in Lemma (2.3).

Lemma 2.5. Making use of the representation of function $R_{\mu,\nu}^k[;]$, we obtain

$$\left(\frac{1}{x}\frac{d}{dx}\right)^m\left[x^{\upsilon}R_{\upsilon,\upsilon}^k(t;\ x,\ a)\right]=t^m x^{\upsilon-m}R_{\upsilon-m,\upsilon}^k(t;\ x,\ a)$$

This follows directly from the differentiation properties of the Bessel functions J_μ and Y_μ [7, Chap. 3].

Theorem 2.2. If $0<m<\frac{1}{2}\upsilon+\frac{3}{4},\ m=1,\ 2,\ \ldots;\ \upsilon>-\frac{1}{2},\ \int_a^\infty\left|x^{m+\frac{1}{2}}f(x)\right|dx<\infty$ and

$$f(t)=W_{\upsilon-m,\upsilon}^k\left[f(s);\ t\right]$$

then

$$f(s)=W_{\upsilon-m,\upsilon}^{k^{-1}}\left[f(t);\ s\right]$$

Proof: By theorem (2.1), we have,

$$x^{-m}F(x)=W_{\upsilon,\upsilon}^{k^{-1}}\left[t^{-m}f(t);\ x\right] \qquad \ldots(2.10)$$

where

$$F(x)=k^{1-\alpha}x^{m-\upsilon}\int_a^x s^{1+\nu-m}\left(x^2+s^2\right)^{\frac{m-1}{2}}J_{m-1}\left[k\left(x^2-s^2\right)^{\frac{1}{2}}\right]f(s)ds$$

Using the definition of operator $W_{\nu,\nu}^{k^{-1}}[;]$, we obtain

$$x^{-m}F(x)=\int_0^\infty\frac{R_{\upsilon,\upsilon}^k(t;\ x,\ a)}{J_\upsilon^2(ta)+Y_\upsilon^2(ta)}t^{1-m}f(t)dt$$

Now, applying the operator $\left(\frac{1}{x}\frac{d}{dx}\right)^m x^\upsilon$ to both the sides of previous equation, we have

$$\left(\frac{1}{x}\frac{d}{dx}\right)^m \left[x^{\upsilon-m}F(x)\right]=\left(\frac{1}{x}\frac{d}{dx}\right)^m (x)^\upsilon \int_0^\infty \frac{R^k_{\upsilon,\upsilon}(t;\, xa)}{J^2_\upsilon(ta)+Y^2_\upsilon(ta)} \, . \, t^{1-m} f(t)dt \qquad \ldots(2.11)$$

Making use of Lemma (2.5) after bringing the differential operator inside the integral sign on the right hand side of above equation and applying (2.9) on left hand side, we obtain

$$f(x)=\int_0^\infty \frac{R^k_{\upsilon-m,\upsilon}(t;\, x,\, a)}{J^2_\upsilon(ta)+Y^2_\upsilon(ta)} tf(t)dt \qquad \ldots(2.12)$$

Bringing of differential operator inside the integral is valid because $\left|R^k_{\nu-m}(t;\, x,\, a)\right|$ is bounded and the resulting integral (2.12) is uniformly convergent for all $x > a$. Thus, the required result is

$$f(x)=W^{k^{-1}}_{\upsilon-m,\upsilon}\left[f(t);\, x\right]$$

SPECIAL CASES

(i) If we set $m=0$ then we obtain results discussed by Nasim [5], Weber and Orr [2, p. 74].

(ii) Setting $k=0$, $\mu=\upsilon-m,\ m=1,2$ gives rise to Weber-Orr transform introduced by Krajeswsk and Obesiak [3].

REFERENCES

1. A. Erdelyi, et.al., Tables of Integral Transforms, Vol. II. McGraw Hill Book Co., Inc., New York, 1954.
2. A. Erdelyi, et. al., Higher Transcendental Functions, Vol. II, Bateman Manuscript Project, McGraw Hill Book Co., Inc., New York, 1953.
3. J. Krajewski and Z. Olesiak, Bull. De L' Academic Polonaise des Sciences Series des Sciences Techniques. XXX No. 7-8, (1982), p. 31-37.
4. J.S. Lowndes, A Generalization of the Erdelyi-Kober Operators, Proc. Edin. Math. Soc. 17 (1970), 139-148.
5. C. Nasim, Associated Weber Integral Transforms of Arbitrary Orders, Indian J. Pure and Appl. Math. 20 (1989), 1126-1138.
6. I.N. Sneddon, A Survey of the use of Operators of Fractional Integration in Applied Mathematics, Research Paper No. 258 (1974). Yellow Series. The University of Calgary, Calgary, Canada.
7. C.N. Watson, A Treatise an the Theory of Bessel Functions, The University Press, Cambridge, England, Second Edition, 1944.