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On some power joined Γ-semigroups

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Abstract: In this paper, we introduce commutative, archimedean, nonpotent Γ-semigroup, Putcha Γ-semigroup and E-m Γ-semigroup. We prove that a finitely generated, commutative, nonpotent, archimedian Γ-semigroup is power joined. We also prove that if S is a commutative, nonpotent, archimedian Γ-semigroup, then, S is power joined if and only if every finitely generalized sub-Γ-semigroup H of S is archimedian. Thereafer, we obtain some equivalent conditions. Then, we prove that an archimedian nonpotent Γ-semigroup is power joined if and only if the structure Γ-group $G_s = S/\rho$ of S is periodic for some $s \in S$; equivalently, it is true for all $s \in S$. We also show that every medial Γ-semigroup S is a left and right Putcha Γ-semigroup and that every E-m Γ-semigroup is a right and left Putcha Γ-semigroup.

Keywords: Γ-idempotent, nonpotent, power joined Γ-semigroup, archimedian Γ-semigroup, periodic Γ-group, medial Γ-semigroup, Putcha Γ-semigroup, E-m Γ-semigroup

AMS Subject Classification: 20M12, 20M10, 20N99, 20N20, 20M17

1 Introduction and Basic Definitions

The idempotent semigroups was introduced by McLean [11] and regular idempotent semigroups was studied by kimura [30]. Radha et al. [12] studied some structures of idempotent commutative semigroup. Rajeswari et al. studied structure of an Idempotent M-Normal Commutative Semigroups[34]. Tamura et al. studied decomposition of a commutative semigroup[47]. Bhambhri et al. studied the concepts of prime and weakly prime left ideals in ternary semiring and gave some characterizations related to the same [46]. Prime and maximal ideals was studied in semigroups [42] and [39]. Tamura defined and studied commutative archimedean semigroups [48]. Ciric et al. [28] studied 0-Archimedian semigroups as a generalization of 0-simple Archimedian semigroups and nil-extensions of 0-simple semigroups. Nagy defined Putcha semigroups [3]. Chrislock defined and studied medial semigroups [14]. McAlister defined power joined semigroup and torsion free semigroup [13]. Bhuniya et al. studied t-Archimedian semigroup and t-Putcha semigroup [2]. Arendt et al. studied the structure of commutative periodic semigroups [10]. Dutta et al. studied periodic Γ-semigroups [44] by generalizing results analogous to those studied by Arendt et al. in [10]. The notion of Γ-semigroups was defined by Sen [17] and then, again by Sen and Saha [18]. Since then, this has been dynamic and potentially active area of research and, as a matter of facts, hundreds of papers have been written by algebraists from across the globe on this subject by generalizing the corresponding results from semigroups and rings to Γ-semigroups [1], [5], [15], [16], [19], [20], [22], [23], [24], [25], [26], [27], [29], [31], [32], [35], [36], [37], [40], [41], [43], [45], [51], [52]. Bhavanari et al. studied some ideal-theoretic results in Γ-nearrings [38]. Recently, Basar et al. studied and derived some results on Γ-semigroups, ordered Γ-semigroups, Γ-hypersemigroups and hypersemigroups [4], [6], [7], [8], [9]. Our results are generalizations of some results using the concepts of Γ-semigroups parallel to those from semigroups in [13], [33] and [49].

The purpose of this paper is to give some basis, and a few basic theorems on a suggested theory of Γ-semigroups. A semigroup is meant a set S closed to a single associative binary operation. A semigroup generalizes a monoid in that there might not exist an identity. It also generalizes a group to a structure in which every element may not have to have an inverse element. On a similar line and pattern, every Γ-semigroup is a generalization of semigroup. A Γ-semigroup is ordered triplets $(S, \Gamma, \cdot)$ consisting of two sets S and Γ and a ternary operation $S \times \Gamma \times S \rightarrow S$ with the property that $(a \cdot x \cdot b) \cdot y \cdot c = a \cdot x \cdot (b \cdot y \cdot c)$ for all $a, b, c \in S$ and $x, y \in \Gamma$. Let A be a nonempty subset of $(S, \Gamma, \cdot)$. Then, A is called a sub-Γ-semigroup of $(S, \Gamma, \cdot)$ if $a \cdot \gamma \cdot b \in A$ for all $a, b \in A$ and $\gamma \in \Gamma$. Furthermore, a Γ-semigroup S is called commutative if $a \cdot \gamma \cdot b = b \cdot \gamma \cdot a$ for all $a, b \in S$ and $\gamma \in \Gamma$. A group S is called Γ-group if $h \cdot \alpha \cdot S = S \cdot \beta \cdot g = S$ for all $(h, g) \in S^2$ and for $\alpha, \beta \in \Gamma$. For subsets A, B of a Γ-semigroup S, the product set $A \cdot B$ of the ordered pair (A, B) relative to S is defined as follows:

$$A \cdot \Gamma \cdot B = \{a \cdot \gamma \cdot b : a \in A, b \in B \text{ and } \gamma \in \Gamma\},$$

and for $A \subseteq S$, the product set $A \cdot A$ relative to S is defined as follows:

$$A^2 = A \cdot A = A \cdot \Gamma \cdot A.$$

Let $A \subseteq S$ and $s \in S$. Then, for a non-negative integer m, the power of A and s is defined as follows:

$$A^m = A \cdot \Gamma \cdot A \cdot \Gamma \cdot A \cdot \Gamma \cdot A \cdots, \text{and } s^m = s \cdot \gamma \cdot s \cdot \gamma \cdot s \cdot \gamma \cdot s \cdots$$

where A occurs m times; Γ *and* γ occur $m - 1$ times.

Note that the power is no more if $m = 0$. Therefore, $A^0 \cdot \Gamma \cdot S = S = S \cdot \Gamma \cdot A^0$ *and* $s^0 \cdot \gamma \cdot s = s = s^0 \cdot \gamma$. That is, S^0 and s^0 act as identity operators. In what follows, we denote the Γ-semigroup $(S, \Gamma, \cdot)$ by S unless otherwise specified. In every part of this paper, for the sake of brevity and typographical compactness, we denote $a \cdot \gamma \cdot b$ by $a\gamma b$. If we consider, $\Gamma = \{1\}$ in the definition of Γ-semigroup, then we notice that every semigroup becomes a particular case of Γ-semigroup, however, every Γ-semigroup is not a semigroup. This makes it ineteresting to comprehend and consider that the results of semigroups in [13], [33] and [49] without Γ become particular case of the corresponding results of Γ-semigroups obtained in this paper and consequently, these results of Γ-semigroup can be obtained for plain semigroups. Examples of Γ-semigroups can be found in [18] and [21].

Example 1.1 *Let S and Γ be two non-empty sets. Fix an element s of S and set $a\gamma b = s$, for all $a, b \in S$ and $\gamma \in \Gamma$. Then, clearly S is a Γ-semigroup.*

Following is an example from Dixit and Dewan [50]:

Example 1.2 *Let $T = \{-i, 0, i\}$ and $\Gamma = T$. Then, T is a Γ-semigroup under the multiplication over complex number while T is not a semigroup under complex number multiplication.*

A Γ-semigroup S is Archimedian if for every $x, y \in S, x^n \in S\Gamma y\Gamma S$ and $y^m \in S\Gamma x\Gamma S$ for some integers n and m. An element s of a Γ-semigroup S is called a Γ-idempotent or simply an idempotent if $s\gamma s = s$ for $\gamma \in \Gamma$. A Γ-semigroup S is idempotent if $S^2 = S\Gamma S = S$.

By analogy with the definitions in plain semigroups, we give the following:

Definition 1.1 *A Γ-semigroup S is called power joined if there exist positive integers m, n such that $h^m = g^n$ for all $h, g \in S$.*

Definition 1.2 *Let S be a Γ-semigroup and $s \in S$. Then, the binary relation ρ_s on S is defined by $x\rho_s y$ if and only if there exists positive integers n and m such that $s^n\gamma x = s^m\gamma y$ for $\gamma \in \Gamma$.*

Note that the relation ρ_s is a congruence relation on S and s is called the standard element determining the corresponding decomposition of S. Furthermore, for any s, S/ρ_s is a Γ-group. The congruence class modulo ρ_s containing s is the identity element of S/ρ_s and it is a sub-Γ-semigroup of S.

Definition 1.3 *Suppose S_α is an arbitrary congruence class of $S(mod\rho)$. Then, the relation $\geq_s$ is a partial order on S_α as follows: $x \geq_s y \Leftarrow y = s^n\gamma x$, or $y = x$, for a positive integer n and $\gamma \in \Gamma$.*

Definition 1.4 *A commutative Γ-semigroup S is called power joined if for every elements a, b, there are positive integers m, n such that $a^m = b^n$.*

Definition 1.5 *A Γ-semigroup S is called a medial Γ-semigroup if it satisfies the identity: $x\alpha a\beta b\gamma y = x\delta b\theta a\lambda y$ for $\alpha, \beta, \gamma, \delta, \theta, \lambda \in \Gamma$.*

Definition 1.6 *A Γ-semigroup S is called a left(right) Putcha Γ-semigroup if for every $x, y \in S$, the hypothesis $y \in x\Gamma S(y \in S\Gamma x)$ implies $y^m \in x^2\Gamma S(y^m \in S\Gamma x^2)$ for some positive integer m. A Γ-semigroup S is called a Putcha Γ-semigroup if for every $x, y \in S$, the assumption $y \in S\Gamma x\Gamma S \Rightarrow y^m \in S\Gamma x^2\Gamma S$ for some positive integer m.*

Definition 1.7 *Let S be a Γ-semigroup, then $\sigma = \{(a, b) \in S \times S: a^n\alpha b = a^{n+1}; a\beta b^n = b^{n+1} for some n \in N\}$.*

Definition 1.8 *Let S be a Γ-semigroup and $\tau = \{(a, b) \in S \times S: a^n = b^n for\ some\ \ n \in N\}$. Then, τ is the finest congruence on S such that S/ρ is torsion free. Furthermore, $\sigma \subseteq \tau$.*

A Γ-semigroup is called nonpotent if it has no Γ-idempotent.

Definition 1.9 *A Γ-semigroup S is called an E-m Γ-semigroup, where m is an integer with $m \geq 2$ if it satisfies the identity: $(a\alpha b)^m = a^m\beta b^m$ for positive integer m and $\alpha, \beta \in \Gamma$.*

Definition 1.10 *An E-m Γ-semigroup for every integer $m \geq 2$ is called an exponential Γ-semigroup.*

For every arbitrary Γ-semigroup S, we denote the set of all positive integers m by $E(S)$for which S obeys the relation: $(a\alpha b)^m = a^m\beta b^m$ and $E(S)$ is called the exponential Γ-semigroup of S.

Definition 1.11 *For a fixed integer $m \geq 2$, a Γ-semigroup S is called an E-m Γ-semigroup if $m \in E(S)$.*

Definition 1.12 *A Γ-semigroup S is called a periodic Γ-semigroup if for any $s \in S$ and any $\gamma \in \Gamma$, there exists positive integers n and m such that $(a)^n \gamma b = (a)^{n+m} \gamma b$ and $b\gamma(a)^n = b\gamma(a)^{n+m}$ for all $b \in M$. Also, a Γ-semigroup S is called a periodic Γ-semigroup if each element of S has a finite order, where the order of $s \in S$ is the order of the cyclic sub-Γ-semigroup of S generated by S, i. e., to each element s of S, for all $\gamma \in \Gamma$, there corresponds an idempotent e and a positive integer n such that $(s)^{n-1} = e$, where s is the Γ-idempotent.*

2 Properties of power joined Γ-semigroups

In this part, we prove that a finitely generated, commutative, nonpotent, archimedian Γ-semigroup is power joined. We also prove that if S is a commutative, nonpotent, archimedian Γ-semigroup, then, S is power joined if and only if every finitely generalized sub-Γ-semigroup H of S is archimedian. Thereafer, we obtain some equivalent conditions. Then, we prove that an archimedian nonpotent Γ-semigroup is power joined if and only if the structure Γ-group $G_s = S/\rho$ of S is periodic for some $s \in S$; equivalently, it is true for all $s \in S$. We also show that every medial Γ-semigroup S is a left and right Putcha Γ-semigroup and that every E-m Γ-semigroup is a right and left Putcha Γ-semigroup. We start with the following:

Theorem 2.1 *Let S be finitely generated, commutative, nonpotent, archimedian Γ-semigroup. Then, S is power joined.*

Proof. Suppose $s \in S$ and consider S/ρ_s. Suppose $s_1, s_2 \in S$ and $\alpha, \beta \in S/\rho_s$ such that $s_1 \in S_\alpha$ and $s_2 \in S_\beta$. As, S/ρ_s is a finite Γ-group, there exists positive integers n and m such that $\alpha^n = \epsilon, \beta^m = \epsilon$, where ϵ is the identity of S/ρ_s. Thus, $s_1^n \in S_\epsilon$ and $s_2^m \in S_\epsilon$. Suppose $\{P_1, P_2, \cdots, P_r\}$ is the set of prime numbers of S which are contained in S_ϵ. Further, suppose that $P = glb\{P_1, P_2, \cdots, P_r\}$, where the partial order in S_ϵ is $\geq_\alpha$. Let $T = \{N: N \in S_\epsilon, N \geq_\alpha P\}$. This set T is finite because S_ϵ is a discrete tree. As S is nonpotent, the powers of s_1^n and s_2^m are all distinct. Thus, there exist positive integers r, t such that $P \geq_\alpha (s_1^n)^r$ and $P \geq_\alpha (s_2^m)^t$. Moreover, for all N, where $P \geq_\alpha N$, there exist positive integers s such that $N = a^s$. So,

$$(s_1^n)^r = a^u, (s_2^m)^t = a^v. \tag{1}$$

and

$$(s_1^{nr})^v = a^u, (s_2^m)^t = a^u. \tag{2}$$

This shows that S is a power joined Γ-semigroup.

Theorem 2.2 *Suppose that S is a commutative, nonpotent, archimedian Γ-semigroup. Then, S is power joined if and only if every finitely generalized sub-Γ-semigroup H of S is archimedian.*

Proof. Suppose that the Γ-semigroup S is power joined. Further, suppose that H is finitely generated sub-Γ-semigroup of S. Then, H is also power joined. Suppose $s_1, s_2 \in H$. Then, there exist positive integers n, m such that $s_1^n = s_2^m$. Let $p = s_1^{m-1}, q = s_2^{n-1}$. We obtain $s_1^n = s_2 \gamma p, s_2^m = s_1 \gamma q$ for some $\gamma \in \Gamma$. The elements p and q are also in H. Let $m = n = 1$, then by arranging the required equations by multiplying both sides of the equation: $s_1^n = s_2^m$ by s_1 or s_2 as desired. Thus, H is archimedian. Suppose $s_1, s_2 \in S$ and H is the sub-Γ-semigroup of S generated by s_1 and s_2. As H is finitely generated, it is archimedian. Thus, H is a finitely generated, commutative, idempotent, archimedian Γ-semigroup, and by Theorem 2.1, we get the desired result that H is power joined. Thus, there exists positive integers n and m such that $s_1^n = s_2^m$. As s_1, s_2 are randomely chosen elements s of S, we can now declare that S is a power joined Γ-semigroup. This completes the proof.

We now obtain the equivalent condition connecting power joined Γ-semigroup, archimedian Γ-semigroup, and its homomorphic image to periodic Γ-semigroup.

Theorem 2.3 *The following conditions on a Γ-semigroup S are equivalent:*

i. The Γ-semigroup S is power joined;

ii. The Γ-semigroup S is archimedian and its Γ-group homomorphic images are periodic;

iii. The Γ-semigroup S satisfies the conditions: for all pairs $a, b \in S$, there are positive integers l, m, n, s, t, p such that

$$a^l = a^n \alpha b^n; b^s = b^t \beta a^p, \tag{3}$$

for $\alpha, \beta \in \Gamma$.

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Suppose S is a power joined Γ-semigroup. Then, obviously, S is an archimedian Γ-semigroup. Suppose G is a Γ-group homomorphic image of S with $\phi: S \to G$ the homomorphism. We will show that G is a periodic Γ-group. Let $a \in G$ and e be the identity of G. There exists $s_1, s_2 \in S$ such that $\phi(s_1) = a\gamma\phi(s_2) = e$ for $\gamma \in \Gamma$. As S is power joined Γ-semigroup, there exists positive integers n, m such that $s_1^n = s_2^m$. Then,

$$\begin{aligned} a^n &= [\phi(s_1)]^n = \phi(s_1^n) \\ &= \phi(s_2^n) \\ &= [\phi(s_2)]^m = e^m = e. \end{aligned}$$

We observe that G is periodic Γ-group, and this completes the proof.

$(i) \Rightarrow (ii)$. Next, we show that $(ii) \Rightarrow (iii)$. Suppose S is an archiedian Γ-semigroup whose Γ-group homomorphic images are periodic.

Case 1. Let e be the Γ-idempotent of S. Then, $S\Gamma e$ is a Γ-group and is the homomorphic image of S. Suppose $a, b \in S$. Then, $a\alpha e$ and $b\beta e$ are elements of $S\Gamma e$. As, $S\Gamma e$ is a periodic Γ-group with e as its identity element, there exists positive integers n and m such that $(a\alpha e)^n = e$ and $(b\beta e)^m = e$. That is, $a^n\alpha e = b^m\beta e$ for $\alpha, \beta \in \Gamma$. Since, S is Archimedian, there exists positive integers k and t and $u, v \in S$ such that

$$a^k = e\alpha v \text{ and } b^t = e\beta u, \tag{4}$$

for $\alpha, \beta \in \Gamma$. From (3) and (4), we obtain the following:

$$a^n\alpha e\beta u = b^m\gamma e\delta u,$$

for some positive integers l, k and m and $\alpha, \beta, \gamma, \delta \in \Gamma$.

or, $a^n\alpha b^t = b^m\gamma b^t$,

or, $a^n\alpha b^t = b^r$, where $r = m + t$.

In a similar fashion, we have $a^l = a^k\alpha b^m$ for some positive integrers l, k and m.

Case 2. Let S do not contain a Γ-idempotent. Suppose $a, b \in S$. Consider the congruence ρ_α. Then, S/ρ_a is a Γ-group homomorphic image of S and thus, it is periodic Γ-group. Moreover, $S = \cup_{\lambda \in S/\rho_a} S_\lambda$ and $a \in S_\epsilon$, where ϵ is the identity of S/ρ_a. So, $\lambda \in S/\rho_a$ such that $b \in S_\lambda$. There exists a positive integer k such that $\lambda^k = \epsilon$. So, $b^k \in S_{\lambda^k} = S_\epsilon$. That is, a and b^k are ρ_a related. By definition of ρ_a, there are positive integers n and m such that $a^n\alpha a = a^m\beta b^k$ for some $\alpha, \beta \in \Gamma$. or, $a^l = a^m\alpha b^k$, where $l = n + 1$. In a similar fashion, we can calculate that $b^s = b^t\alpha a^p$. Hence, $(ii) \Rightarrow (iii)$.

We now show, $(iii) \Rightarrow (i)$.

Case 1.

$$e^l = e^m\alpha a^n, a^s = a^t\beta e^p. \tag{5}$$

$$e = e\alpha a^n, a^s = a^t\alpha e. \tag{6}$$

By the relation (6), we have the following:

$$\begin{aligned} e = e^t &= (e\alpha a^n)^t \\ &= e^t\beta(a^t)^n \\ &= (e\gamma a^t)^n = (a^s)^n, \end{aligned}$$

for $\alpha, \beta, \gamma \in \Gamma$. Therefore, we have $e = a^r$ for a positive inteher r. It is now clear that if $a, b \in S$, there are positive integers u and v such that $a^u = b^v$. Hence, S is power joined Γ-semigroup.

Case 2. Suppose S has no Γ-idempotent. Again, we have for any pair $a, b \in S$, positive integers l, m, n, s, t and p such that

$$a^l = a^m\alpha b^n \text{and } b^s = b^t\beta a^p, \tag{7}$$

for $\alpha, \beta \in \Gamma$. We will prove that there are positive integers l' and n' such that $a^{l'} = a^m\alpha b^{n'}$ and $n'\alpha p \geq mt$. As, S does not have a Γ-idempotent, ł $> m$ in (7). Then, we have the following:

$$\begin{aligned} a^{2l-m} &= a^{l-m}\alpha a^l = a^{l-m}\beta a^m\gamma b^n \\ &= a^l\delta b^n \\ &= (a^m\lambda b^n)\theta b^n \\ &= a^m\theta b^{2n} \end{aligned}$$

for $\alpha, \beta, \gamma, \theta, \lambda, \delta \in \Gamma$. Now, suppose that for some integer $k \geq 1$, we have

$$a^{kl-(k-m)m} = a^m\beta b^{kn}.$$

We will prove that

$$a^{(k+1)l-km} = a^m\alpha b^{(k+1)n}.$$

Now, we have the following:

$$\begin{aligned} a^{(k+1)l-km} &= a^{kl-km}\alpha a^l = a^{kl-km}\beta(a^m\gamma b^n) \\ &= (a^{kl-km}\theta a^m)\delta b^n \\ &= a^{kl-(k-1)m}\gamma_1 b^n \\ &= (a^m\gamma_2 b^{kn})\gamma_3 b^n \\ &= a^m\gamma_4 b^{(k+1)n} \\ &= a^{kl-(k-1)m}\gamma_5 b^n \\ &= (a^m\gamma_6 b^{kn})\gamma_7 b^n \\ &= a^m\gamma_8 b^{(k+1)n} \end{aligned}$$

for $\alpha, \beta, \gamma, \delta, \theta, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7 \in \Gamma$.

By induction, we obtain for every $k \geq 1$,

$$e^{kl-(k-1)m} = a^m\alpha b^{kn},$$

for $\alpha \in \Gamma$.

Now, choose k such that $knp \geq mt$. Set $n' = kn$, $l' = kl - (k-1)m$. We replace the equations of (7) by the following:

$$a^{l'} = a^m \alpha b^{n'} \text{ and } b^s = b^t \beta^p, \quad (8)$$

for $\alpha, \beta \in \Gamma$.

From (8), we calculate the following:

$$a^{l'tp} = (b^{n'})^{tp}(a^m)^{tp} = (b^t)^{n'p}(a^p)^{mt},$$

or,

$$a^{l'tp} = (b^t)^{mt+(n'p-mt)}\alpha(a^p)^{mt} = (b^t)^{mt}\beta(b^t)^{n'p-mt}\gamma_1(a^p)^{mt}$$
$$= (b^t a^p)^{mt}\gamma_2(b^t)^{n'p-mt}$$
$$= (b^8)^{mt}\gamma_3(b^t)^{n'p-mt}$$

for $\alpha, \beta, \gamma_1, \gamma_2, \gamma_3 \in \Gamma$. Set $u = l'tp$ and $v = smt + t(n'p - mt)$. We see that we have obtained the equation $a^u = b^v$. Hence, S is power joined and thus, we have shown $(iii) \Rightarrow (i)$.

In the following Proposition, we prove further conditions equivalent to those equivalent conditions in the Theorem 2.3.

Proposition 2.1 *Each of (i), (ii) and (iii) in the Theorem 2.3 is equivalent to one of (iv) and (v) below.*

iv. The Γ-semigroup S satisfies the following conditions: there is an element a_0 of S such that for all $b \in S$, there are positive integers l, m, n, s, t, p obeying $a_0^l = a_0^m \alpha b^n$ and $b^\epsilon = b^t \beta a_0^p$ for $\alpha, \beta \in \Gamma$.

v. The Γ-semigroup S satisfies the condition: for all $a, b \in S$, there are positive integers l, m, n, s, t such that $a^l = (a\alpha b)^m$ and $b^s = (b\beta a)^t$ for $\alpha, \beta \in \Gamma$.

Proof. We define a relation τ on S as follows:

$$a\tau b \Leftarrow a^l = a^m \alpha b^n \text{ and } b^s = b^t \beta a^p$$

for $\alpha, \beta \in \Gamma$ and for some l, m, n, s, t, p. Then, τ is an equivalence relation on S. Reflexivity and symmetry are obvious. Transitivity is proved as follows:

Suppose $a^l = a^m \alpha b^n$ and $b^k = b^q \beta c^p$ for $\alpha, \beta \in \Gamma$. First, we have the following:

$a^{lk} = a^{mk}\alpha b^{nk} = a^{mk}\beta b^{nq}\gamma c^{nv}$ for α, β, γ and then, $a^{l'} = a^{m'}\alpha(a^{mq}\beta b^{nq}\gamma c^{nv}) = a^{m'+lq}\delta c^{nv}$ for $\alpha, \beta, \gamma, \delta \in \Gamma$, where $l' = lk, m' = mk - mq$ if $k \geq q$, $l' = lk = mq - mk, m' = 0$ if $k < q$.

$(iv) \Rightarrow (iii)$. It is obtained as an immediate consequence.

$(iii) \Rightarrow (iv)$, $(i) \Rightarrow (v)$, $(v) \Rightarrow (iii)$. are straightforward.

The following Theorem gives us the necessary and sufficient condition as to when a nonpotent, archimedian Γ-semigroup becomes a power joined Γ-semigroup.

Theorem 2.4 *An archimedian Γ-semigroup without Γ-idempotent is power joined if and only if the structure Γ-group $G_s = S/\rho$ of S is periodic for some $s \in S$, equivalently for all $s \in S$.*

Proof. Suppose Γ-semigroup $S$is an archmedian Γ-semigroup without Γ-idempotent. Then, the statement (iii) of Theorem 2.3 is equivalent to:

S/ρ_s is periodic for all $s \in S$. Theorem 2.3 (iv) is equivalent to the following:

S/ρ_s is periodic or some $s \in S$. The first stament is obvious. To see the second, we will prove the following:

If S/ρ_{a_0} is periodic, then for all $b \in S$, there are positive integers l, m, n, s, t, p such that

$$a_0^l = a_0^m \alpha b^n, b^n = b^t \gamma a_0^p, \quad (9)$$

for $\alpha, \gamma \in \Gamma$. The first of (9) is immediately obtained. As S is archimedian, there is a positive integer k and an element $c \in S$ such that $b^k = a_0 \alpha c$ for $\alpha \in \Gamma$, which implies $b^{kl} = a_0^l \alpha c^l$ for $\alpha \in \Gamma$. Since, S has no Γ-idempotent, $l > m$ in the first of (9). Now, we have the following:

$$b^{kl}\alpha a_0^{l-m} = a_0^{l-m}\beta a_0^l \gamma c^l = a_0^{l-m}\delta a_0^m \lambda b^n \gamma_1 c^l$$
$$= b^n \gamma_2 b^{kl}$$
$$= b^{n+kl}.$$

for $\alpha, \beta, \gamma, \delta, \gamma_1, \gamma_2 \in \Gamma$. This completes the proof.

The following Theorem gives us information about the connection between a medial Γ-semigroup and a Putcha Γ-semigroup.

Theorem 2.5 *Every medial Γ-semigroup S is a left and right Putcha Γ-semigroup.*

Proof. Suppose S is a medial Γ-semigroup and $a, b \in S$ be arbitrary elements with $b \in a\Gamma S$. Therefore, $b = a\gamma x$ for some $x \in S$ and $\gamma \in \Gamma$. Then, $b^2 = (a\alpha x)^2 = a^2\gamma x^2$ for $\alpha, \gamma \in \Gamma$. Thus, $b^2 \in a^2\Gamma S$. Hence, S is a left Putcha Γ-semigroup. In a similar fashion, we can prove that S is a right Putcha Γ-semigroup.

Corollary 2.1 *Suppose S is a Γ-semigroup. Then, the following are equivalent:*

i. S has a Γ-idempotent;

ii. S/σ has a Γ-idempotent;

iii. S/τ has a Γ-idempotent. If, further, S is archimedian, then the conditions (i), (ii) and (iii) are equivalent to the condition (iv) below:

iv. There exists $x, y \in S$ with $x = x\alpha y$ for some $\alpha \in \Gamma$.

Proof. Clearly, we have $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$.

Now, we prove $(iii) \Rightarrow (i)$. Suppose that the given condition (iii) is satisfied and let $x \in S$ be such that $(x^2, x) \in \tau$. Then, $x^n = (x^2)^n = (x^n)^2$ for some $n \in N$, where N is a set of positive integers. Hence, x^n is Γ-idempotent.

$(i) \Rightarrow (iv)$. For any Γ-semigroup S has a Γ-idempotent e, there exists $x(= e), y(= e)$ such that $x = x\alpha y$ for some $\alpha \in \Gamma$. Hence, $(i) \Rightarrow (iv)$.

$(iv) \Rightarrow (ii)$. If S is an archimedian Γ-semigroup, then S/σ is cancellative and that $x = x\alpha y$ for $\alpha \in \Gamma$ gives us $(y, y^2) \in \sigma$. Therefore, S/σ has a Γ-idempotent. Hence, $(iv) \Rightarrow (ii)$.

Theorem 2.6 *If a Γ-semigroup S obeys the condition: $(a\alpha b)^2 = a^2\beta b^2$, then it also obeys the relation: $(a\alpha b)^n = a^2\beta b^n$ for $\alpha, \beta \in \Gamma$ and for all positive integers $n \geq 4$.*

Proof. As $2 \in E(S) \Rightarrow 4 \in E(S)$. Therefore, it is sufficient to prove that if $n > 2$ and $2, n \in E(S)$, then $n + 1 \in E(S)$. Let $2, n \in E(S)$for an integer $n > 2$. Suppose that n is odd. Thus, there is a positive integer$k$ such that $n - 1 = 2k$ and, for $a, b \in S$, we have the following relation:

$$\begin{aligned}
a^{n+1}\alpha b^{n+1} &= a\beta(a^n\delta b^n)\theta b = a\gamma_1(a\gamma_2 b)^n\gamma_3 b \\
&= a^2\gamma_4(b\gamma_5 a)^{n-1}\gamma_6 b^2 \\
&= a^2\gamma_7((b\gamma_8 a)^k)^2\gamma_9 b^2 \\
&= (a\gamma_{10}(b\gamma_{11} a)^k)^2\gamma_{12} b^2 \\
&= ((a\gamma_{13})^k\gamma_{14} a)^2\gamma_{15} b^2 \\
&= (a\gamma_{16} b)^{2k}\gamma_{17} a^2\gamma_{18} b^2 \\
&= (a\gamma_{19} b)^{n-1}\gamma_{20}(a\gamma_{21} b)^2 \\
&= (a\gamma_{22} b)^{n+1}.
\end{aligned}$$

for$\alpha, \beta, \delta, \theta, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{15}, \gamma_{16}, \gamma_{17}, \gamma_{18}, \gamma_{19}, \gamma_{20}, \gamma_{21}, \gamma_{22} \in \Gamma$.

Furthermore, we suppose that n is even. Thus, there exists a positive integer k such that $n - 2 = 2k$ and, for arbitrary $a, b \in S$, we have the following:

$$\begin{aligned}
a^{n+1}\gamma_1 b^{n+1} &= a\gamma_2(a^n\gamma_3 b^n)\gamma_4 b \\
&= a\gamma_5(a\gamma_6 b)^n\gamma_7 b \\
&= a^2\gamma_8(b\gamma_9 a)^{n-1}\gamma_{10} b^2 \\
&= a^2\gamma_{10}(b\gamma_{11} a)^{n-3}\gamma_{12}(b\gamma_{13} a)^2\gamma_{14} b^2 \\
&= a^2\gamma_{15}(b\gamma_{16} a)^{n-2}\gamma_{17}(b\gamma_{18} a\gamma_{19} b)^2 \\
&= a^2\gamma_{20}(b\gamma_{21} a)^{n-2}\gamma_{22} b^2\gamma_{23} a\gamma_{24} b \\
&= a^2\gamma_{25}((b\gamma_{26} a)^k)^2\gamma_{27} b^2\gamma_{28} a\gamma_{29} b \\
&= (a\gamma_{30}(b\gamma_{31} a)^k)^2\gamma_{32} b^2\gamma_{33} a\gamma_{34} b \\
&= ((a\gamma_{35} b)^k\gamma_{36} a)^2\gamma_{37} b^2\gamma_{38} a\gamma_{39} b \\
&= (a\gamma_{40} b)^{2k}\gamma_{41} a^2\gamma_{42} b^2\gamma_{43} a\gamma_{44} b \\
&= (a\gamma_{45} b)^{n-2}\gamma_{46}(a\gamma_{47} b)^2\gamma_{48}(a\gamma_{49} b) \\
&= (a\gamma_{50} b)^{n+1}.
\end{aligned}$$

for $\gamma_i \in \Gamma, i = 1,2,\cdots,50$.

Corollary 2.2 *A Γ-semigroup S is exponential if and only if it obeys the following condition:*

$$(a\alpha b)^2 = a^2\beta b^2 \, and (a\alpha b)^3 = a^3\beta b^3,$$

for $\alpha, \beta \in \Gamma$.

Proof. As a consequence of Theorem 2.6.

Corollary 2.3 *Suppose S is an E-2 Γ-semigroup. Then, either $E(S) = N($ and thus, S is an exponential Γ-semigroup) or $E(S) = N - \{3\}$, where N denotes the set of all positive integers.*

Proof. As a consequence of Theorem 2.6.

Theorem 2.7 *Every E-m Γ-semigroup is a right and left Putcha Γ-semigroup.*

Proof. Suppose S is an E-m Γ-semigroup for some m. Suppose $a, b \in S$ with $b \in a\Gamma S$. Therefore, $b = a\gamma s$ for some $s \in S$ and $\gamma \in \Gamma$. Thus, $b^m = (a\alpha s)^m = a^m\beta b^m \in a^m\Gamma S$. Hence, S is a left Putcha Γ-semigroup. In a similar fashion, S is a right Putcha Γ-semigroup.

Corollary 2.4 *Every exponential Γ-semigroup is a right and left Putcha Γ-semigroup.*

Proof. Obvious by Theorem 2.7.

Theorem 2.8 *Every E-m Γ semigroup is a semilattice of archimedian E-m Γ-semigroup.*

Proof. It is a consequence of Theorem 2.7.

Conclusion: Evidently, from some literature referred to in the bibliography of this paper, it is amply clear that several classical concepts and properties of the theory of semigroups and rings have been generalized and extended by the scientists. Motivated by this, it is ideally and naturally tempting to generalize the results from semigroups to Γ-semigroups as Γ-semigroups is a generalization of semigroups. In this paper, we investigated commutative, archimedean, nonpotent Γ-semigroups and obtained some results. We proved that a finitely generated, commutative, nonpotent, archimedian Γ-semigroup is power joined. We also proved that if S is a commutative, nonpotent, archimedian Γ-semigroup, then, S is power joined if and only if every finitely generalized sub-Γ-semigroup H of S is archimedian. Thereafer, we obtained some equivalent conditions. Then, we proved that an archimedian nonpotent Γ-semigroup is power joined if and only if the structure Γ-group $G_s = S/\rho$ of S is periodic for some $s \in S$; equivalently, it is true for all $s \in S$. Finally, we showed that every medial Γ-semigroup S is a left and right Putcha Γ-semigroup. The results can further be generalized and axtended to other algebraic structures.

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