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LIMITING IMBEDDINGS OF FRACTIONAL SOBOLEV SPACES

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ABSTRACT :

In this paper we give overview results about the Bourgain, Brezis, and Mironescu theorem regarding limiting embeddings of fractional Sobolev spaces. We give this theorem on the asymptotic behaviour of the norm of the Sobolev-type embedding operator : $\mathcal{W}_0^{1-\varepsilon,1+\varepsilon} \to L^{n(1+\varepsilon)/(n-(1-\varepsilon^2))}$ as $\varepsilon \uparrow 0$ and $1-\varepsilon \uparrow n/(1+\varepsilon)$. We extended results of this theorem for all values of $1-\varepsilon \in (0,1)$.

KEYWORDS:

embeddings,
Sobolev spaces,
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INTRDUCTION

If $(1-\varepsilon) \in (0,1)$ and $\varepsilon \geq 0$.We have the space $\mathcal{W}_0^{1-\varepsilon,1+\varepsilon}(\mathrm{R}^n)$ as the completion of $C_0^\infty(\mathrm{R}^n)$ in the norm $\left(\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{|u(x)-u(y)|^{1+\varepsilon}}{|x-y|^{n+(1-\varepsilon)(1+\varepsilon)}}dx\,dy\right)^{1/(1+\varepsilon)}$. We also need the space

$\mathcal{W}_{\perp}^{1-\varepsilon,1+\varepsilon}(\mathrm{Q})$ of functions defined on the cube $\mathrm{Q} = \{\mathrm{x} \in \mathrm{R}^{\mathrm{n}} : |\mathrm{x}_{\mathrm{i}}| < 1/2, 1 \le i \le n\}$ which are orthogonal to 1 with the finite norm

$$\left(\int\limits_{Q}\int\limits_{Q}\frac{|u(x)-u(y)|^{1+\varepsilon}}{|x-y|^{n+(1-\varepsilon)(1+\varepsilon)}}dx\,dy\right)^{1/(1+\varepsilon)}.$$

The main result by Bourgain et al. [2][3] is the inequality

$$\|u\|_{L^{q}(Q)}^{1+\varepsilon} \le c(n)\frac{\varepsilon}{(n-(1-\varepsilon^2))^{\varepsilon}}\|u\|_{\mathcal{W}_{\perp}^{1-\varepsilon,1+\varepsilon}(Q)}^{1+\varepsilon}, \tag{1}$$

where $u \in \mathcal{W}_{\perp}^{1-\varepsilon,1+\varepsilon}(\mathrm{Q}), 0 < \varepsilon \le \frac{1}{2}, 1-\varepsilon^2 < n,\ q = n(1+\varepsilon)/(n-(1-\varepsilon^2))$ and $c(n)$ depends on n.

The present article is a direct outgrowth of this result. Figuring out a similar estimate for functions in $\mathcal{W}_{0}^{1-\varepsilon,1+\varepsilon}(\mathrm{R}^{\mathrm{n}})$, valid for the whole interval $0 < \varepsilon < 1$, one could anticipate the appearance of the factor $\varepsilon(1-\varepsilon)$ in the right-hand side, since the norm in $\mathcal{W}_{0}^{1-\varepsilon,1+\varepsilon}(\mathrm{R}^{\mathrm{n}})$ blows up both as $\varepsilon \uparrow 2$ and $\varepsilon \downarrow 1$, firstly we give Hardy-type inequalities.

Theorem 1. Let $n \ge 1$, $0 \le \varepsilon < 1$, and $1-\varepsilon^2 < n$. Then, for an arbitrary function $u \in \mathcal{W}_{0}^{1-\varepsilon,1+\varepsilon}(\mathrm{R}^{\mathrm{n}})$, there holds

$$\int\limits_{\mathbf{R}^n}|u(x)|^{1+\varepsilon}\frac{dx}{|x|^{1-\varepsilon^2}} \le c(n,1+\varepsilon)\frac{\varepsilon(1-\varepsilon)}{(n-(1-\varepsilon^2))^{1+\varepsilon}}\|u\|_{\mathcal{W}_0^{1-\varepsilon,1+\varepsilon}(\mathbf{R}^n)}^{1+\varepsilon}. \tag{2}$$

Proof. Let

$$\psi(h) = |(1-\varepsilon)^{n-1}|^{-1}n(n+1)(1-|h|)_+,$$

where $h \in \mathrm{R}^{\mathrm{n}}$ and plus stands for the nonnegative part of a real-valued function.

We introduce the standard extension of u onto

$$\mathrm{R}_+^{n+1} = \{(x,z) : x \in \mathrm{R}^n, z > 0\}U(x,z) := \int\limits_{\mathbf{R}^n}\psi(h)u(x+zh)dh.$$

A routine majoration implies $|\nabla U(x,z)| \le \frac{n(n+1)(n+2)}{z|(1-\varepsilon)^{n-1}|}\int_{|h|<1}|u(x+zh)-u(x)|dh.$

Hence and by Hölder's inequality one has

$$\int\limits_0^\infty\int\limits_{\mathbf{R}^n} z^{-1+\varepsilon(1+\varepsilon)}|\nabla U(x,z)|^{1+\varepsilon}dx\,dz \le \frac{n}{|(1-\varepsilon)^{n-1}|}(n+1)^{1+\varepsilon}(n+2)^{1+\varepsilon}$$

$$\times\int\limits_0^\infty z^{(\varepsilon^2-2)}\int\limits_{|h|<1}\int\limits_{\mathbf{R}^n}|u(x+zh)-u(x)|^{1+\varepsilon}dx\,dh\,dz. \quad (3)$$

Setting $\eta = zh$ and changing the order of integration, one can rewrite (3) as

$$\int_0^\infty \int_{\mathbf{R}^n} z^{-1+\varepsilon(1+\varepsilon)} |\nabla U(x,z)|^{1+\varepsilon} dx\, dz$$

$$\leq \frac{n(n+1)^{1+\varepsilon}(n+2)^{1+\varepsilon}}{|(1-\varepsilon)^{n-1}|((1-\varepsilon^2)+n)} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{|u(x)-u(y)|^{1+\varepsilon}}{|x-y|^{n+(1-\varepsilon^2)}} dx\, dy. \quad (4)$$

By Hardy's inequality,

$\int_0^{|x|} z^{(\varepsilon^2-2)} \left|\int_0^z \varphi(\tau)\, dt\right|^{1+\varepsilon} dz \leq (1-\varepsilon)^{-(1+\varepsilon)} \int_0^{|x|} z^{-1+\varepsilon(1+\varepsilon)} |\varphi(z)|^{1+\varepsilon}\, dz$ one has

$$\frac{|u(x)|^{1+\varepsilon}}{|x|^{1-\varepsilon^2}} = \varepsilon(1+\varepsilon) \int_0^{|x|} z^{-1+\varepsilon(1+\varepsilon)} dz \frac{|u(x)|^{1+\varepsilon}}{|x|^{1+\varepsilon}}$$

$$\leq \varepsilon(1+\varepsilon) \int_0^{|x|} z^{(\varepsilon^2-2)} dz \left(\int_0^z \left(\left|\frac{\partial U}{\partial \tau}(x,\tau)\right| + \frac{|U(x,\tau)|}{|x|} \right) d\tau \right)^{1+\varepsilon}$$

$$\leq \frac{\varepsilon(1+\varepsilon)}{(1-\varepsilon)^{1+\varepsilon}} \int_0^{|x|} z^{-1+\varepsilon(1+\varepsilon)} \left(\left|\frac{\partial U}{\partial z}(x,z)\right| + \frac{U(x,z)}{|x|} \right)^{1+\varepsilon} dz\,.$$

Now, the integration over R^n and Minkowski's inequality imply

$$\int_{\mathbf{R}^n} \frac{|u(x)|^{1+\varepsilon}}{|x|^{1-\varepsilon^2}} dx$$

$$\leq \frac{\varepsilon(1+\varepsilon)}{(1-\varepsilon)^{1+\varepsilon}} \left(\left(\int_{\mathbf{R}^n} \int_0^\infty z^{-1+\varepsilon(1+\varepsilon)} \left|\frac{\partial U}{\partial z}(x,z)\right|^{1+\varepsilon} dz\, dx \right)^{1/(1+\varepsilon)} + A \right)^{1+\varepsilon}, \quad (5)$$

Where $A := \left(\int_{\mathbf{R}^n} \int_0^{|x|} z^{-1+\varepsilon(1+\varepsilon)} |x|^{-(1+\varepsilon)} |U(x,z)|^{1+\varepsilon}\, dz\, dx \right)^{1/(1+\varepsilon)}$.

Clearly, $A^{1+\varepsilon} \leq 2^{(1+\varepsilon)/2} \int_{\mathbf{R}^n} dx \int_0^\infty z^{-1+\varepsilon(1+\varepsilon)} \frac{|U(x,z)|^{1+\varepsilon}}{(x^2+z^2)^{(1+\varepsilon)/2}} dz\, dx$, which does not exceed

$$2^{(1+\varepsilon)/2} \int_{(1-\varepsilon)^n_+} (\cos\theta)^{-1+\varepsilon(1+\varepsilon)} \int_0^\infty |U|^{1+\varepsilon} \rho^{(n+\varepsilon^2-2)}\, d\rho\, d\sigma, \quad (6)$$

where $\rho = (x^2+z^2)^{1/2}, \cos\theta = z/\rho, d\sigma$ is an element of the surface area on the unit sphere $(1-\varepsilon)^n$, and $(1-\varepsilon)^n_+$ is the upper half of $(1-\varepsilon)^n$.

Using Hardy's inequality

$\int_0^\infty |U|^{1+\varepsilon}\rho^{(n+\varepsilon^2-2)}\,d\rho \le \left(\frac{1+\varepsilon}{n-(1-\varepsilon^2)}\right)^{1+\varepsilon}\int_0^\infty \left|\frac{\partial U}{\partial\rho}\right|^{1+\varepsilon}\rho^{n-1+\varepsilon(1+\varepsilon)}\,d\rho$,one arrives at the estimate

$$A^{1+\varepsilon} \le \left(\frac{2^{\frac{1}{2}}(1+\varepsilon)}{n-(1-\varepsilon^2)}\right)^{1+\varepsilon}\int_0^\infty\int_{\mathbf{R}^n} z^{-1+\varepsilon(1+\varepsilon)}|\nabla U(x,z)|^{1+\varepsilon}\,dx\,dz\,.$$

Combining this with (5), one obtains

$$\int_{\mathbf{R}^n}\frac{|u(x)|^{1+\varepsilon}}{|x|^{1-\varepsilon^2}}\,dx \le \frac{\varepsilon(1+\varepsilon)}{(1-\varepsilon)^{1+\varepsilon}}\left(1+\frac{2^{\frac{1}{2}}(1+\varepsilon)}{n-(1-\varepsilon^2)}\right)^{1+\varepsilon}\int_0^\infty\int_{\mathbf{R}^n} z^{-1+\varepsilon(1+\varepsilon)}|\nabla U(x,z)|^{1+\varepsilon}dx\,dz$$

which, along with (5), gives

$$\int_{\mathbf{R}^n}\frac{|u(x)|^{1+\varepsilon}}{|x|^{1-\varepsilon^2}}\,dx$$

$$\le \frac{\varepsilon}{(n-(1-\varepsilon^2))^{1+\varepsilon}}\frac{(1+\varepsilon)(n+2(1+\varepsilon))^{3(1+\varepsilon)}}{|(1-\varepsilon)^{n-1}|(1-\varepsilon)^{1+\varepsilon}}\|u\|^{1+\varepsilon}_{\mathcal{W}_0^{1-\varepsilon,1+\varepsilon}(\mathbf{R}^n)}. \qquad (7)$$

In order to justify (2) we need to improve (2) for small values of $(1-\varepsilon)$.

Clearly, $\frac{|(1-\varepsilon)^{n-1}|}{2^{(1-\varepsilon^2)}(1-\varepsilon^2)}\int_{\mathbf{R}^n}\frac{|u(x)|^{1+\varepsilon}}{|x|^{1-\varepsilon^2}}\,dx = \int_{\mathbf{R}^n}\int_{|x-y|>2|x|}\frac{dy}{|x-y|^{n+(1-\varepsilon^2)}}|u(x)|^{1+\varepsilon}\,dx.$

Since $|x-y|>2|x|$ implies $2|y|/3<|x-y|<2|y|$, we obtain

$$\left(\frac{|(1-\varepsilon)^{n-1}|}{2^{(1-\varepsilon^2)}(1-\varepsilon^2)}\int_{\mathbf{R}^n}\frac{|u(x)|^{1+\varepsilon}}{|x|^{1-\varepsilon^2}}\,dx\right)^{1/(1+\varepsilon)}$$

$$\le\left(\int_{\mathbf{R}^n}\int_{|x-y|>|x|}\frac{|u(x)-u(y)|^{1+\varepsilon}}{|x-y|^{n+(1-\varepsilon^2)}}\,dx\,dy\right)^{1/(1+\varepsilon)}$$

$$+\left(|(1-\varepsilon)^{n-1}|\frac{3^{(1-\varepsilon^2)}-1}{2^{(1-\varepsilon^2)}(1-\varepsilon^2)}\int_{\mathbf{R}^n}\frac{|u(y)|^{1+\varepsilon}}{|y|^{1-\varepsilon^2}}\,dy\right)^{1/(1+\varepsilon)}.$$

Hence,

$$\left(\frac{|(1-\varepsilon)^{n-1}|}{2^{(1-\varepsilon^2)}(1-\varepsilon^2)}\right)^{1/(1+\varepsilon)}\left(1-\left(3^{(1-\varepsilon^2)}-1\right)^{1/(1+\varepsilon)}\right)\left(\int_{\mathbf{R}^n}\frac{|u(x)|^{1+\varepsilon}}{|x|^{1-\varepsilon^2}}\,dx\right)^{1/(1+\varepsilon)}$$

$$\le 2^{-1/(1+\varepsilon)}\|u\|_{\mathcal{W}_0^{1-\varepsilon,1+\varepsilon}(\mathbf{R}^n)}.$$

Let δ be an arbitrary number in $(0,1)$. If $(1-\varepsilon)\le(4(1+\varepsilon))^{-1}\delta^{1+\varepsilon}$, we conclude

$$\int_{\mathbf{R}^n} \frac{|u(x)|^{1+\varepsilon}}{|x|^{1-\varepsilon^2}} dx \le \frac{2^{(1-\varepsilon^2)-1}(1-\varepsilon^2)}{|(1-\varepsilon)^{n-1}|(1-\delta)^{1+\varepsilon}} \|u\|^{1+\varepsilon}_{\mathcal{W}_0^{1-\varepsilon,1+\varepsilon}(\mathbf{R}^n)}. \quad (8)$$

Setting $\delta = 2^{-1}$ and comparing this inequality with (6), we arrive at (2) with

$c(n,(1+\varepsilon)) = |(1-\varepsilon)^{n-1}|^{-1}(n+2(1+\varepsilon))^{3(1+\varepsilon)}(1+\varepsilon)^{3+\varepsilon}2^{(n+1)(n+2)}$.

The proof is complete.

From Theorem 1. we shall deduce an inequality, analogous to (2), for functions defined on the cube Q. Unlike (3), this inequality contains no factor s in the right-hand side, which is not surprising, because, for smooth u, the norm $\|u\|_{\mathcal{W}_\perp^{1-\varepsilon,1+\varepsilon}(Q)}$ tends to a finite limit as $\varepsilon \downarrow 1$.

Corollary 2. Let $n \ge 1$, $0 \le \varepsilon < 1$, and $(1-\varepsilon^2) < n$. Then any function $u \in \mathcal{W}_\perp^{1-\varepsilon,1+\varepsilon}(Q)$ satisfies

$$\int_Q |u(x)|^{1+\varepsilon} \frac{dx}{|x|^{1-\varepsilon^2}} \le c(n,(1+\varepsilon)) \frac{\varepsilon}{(n-(1-\varepsilon^2))^{1+\varepsilon}} \|u\|^{1+\varepsilon}_{\mathcal{W}_\perp^{1-\varepsilon,1+\varepsilon}(Q)}. \quad (9)$$

Proof. Let us preserve the notation u for the mirror extension of $u \in \mathcal{W}_\perp^{1-\varepsilon,1+\varepsilon}(Q)$ to the cube 3Q, where aQ stands for the cube obtained from Q by dilation with the coefficient a. We choose acut-off function η, equal to 1 on Q and vanishing outside 2Q, say, $\eta(x) = \prod_{i=1}^n \min\left\{1, 2\left(1-x_i\right)_+\right\}$. By Theorem 1, it is enough to prove that

$$\|\eta u\|^{1+\varepsilon}_{\mathcal{W}_0^{1-\varepsilon,1+\varepsilon}(\mathbf{R}^n)} \le (1-\varepsilon)^{-1} c(n,(1+\varepsilon)) \|u\|^{1+\varepsilon}_{\mathcal{W}_\perp^{1-\varepsilon,1+\varepsilon}(Q)}. \quad (10)$$

Clearly, the norm in the left-hand side is majorized by

$$\left(\int_{3Q}\int_{3Q} \frac{|u(x)-u(y)|^{1+\varepsilon}}{|x-y|^{n+(1-\varepsilon^2)}} dx\, \eta(y)^{1+\varepsilon} dy\right)^{1/(1+\varepsilon)}$$

$$+\left(\int_{3Q}\int_{3Q} \frac{|\eta(x)-\eta(y)|^{1+\varepsilon}}{|x-y|^{n+(1+\varepsilon^2)}} dx |u(y)|^{1+\varepsilon} dy\right)^{1/(1+\varepsilon)}$$

$$+\left(2\int_{3Q}\int_{\mathbf{R}^n\setminus 3Q} \frac{dy}{|x-y|^{n+(1-\varepsilon^2)}} |(\eta u)(x)|^{1+\varepsilon} dx\right)^{1/(1+\varepsilon)}.$$

The first term does not exceed $6^{n/(1+\varepsilon)}\|u\|_{\mathcal{W}_\perp^{1-\varepsilon,1+\varepsilon}(Q)}$; the second term is not greater than

$$2n^{1/2}\left(\int_{3Q}\int_{3Q}\frac{dy}{|x-y|^{n-\varepsilon}}|u(y)|^{1+\varepsilon}\,dy\right)^{1/(1+\varepsilon)}$$

$$\leq n3^{2+n/(1+\varepsilon)}\left(\frac{|(1-\varepsilon)^{n-1}|}{\varepsilon(1+\varepsilon)}\right)^{1/(1+\varepsilon)}\|u\|_{L^{1+\varepsilon}(Q)},$$

and the third one is dominated by

$$\left(2\int_{2Q}\int_{|x-y|>1/2}\frac{dy}{|x-y|^{n+(1-\varepsilon^2)}}|u(x)|^{1+\varepsilon}\,dx\right)^{1/(1+\varepsilon)}\leq\left(\frac{2^{n+2+\varepsilon}}{sp}\right)^{1/(1+\varepsilon)}\|u\|_{L^{1+\varepsilon}(Q)}.$$

Summing up these estimates, one obtains

$$\|\eta u\|_{\mathcal{W}_0^{1-\varepsilon,1+\varepsilon}(\mathbf{R}^n)}\leq 6^{\frac{n}{1+\varepsilon}}\|u\|_{\mathcal{W}_0^{1-\varepsilon,1+\varepsilon}(Q)}$$

$$+n3^{2+n/(1+\varepsilon)}(1+\varepsilon)^{-1/(1+\varepsilon)}\big((1-\varepsilon)^{-1/(1+\varepsilon)}+\varepsilon^{-1/(1+\varepsilon)}\big)\|u\|_{L^{1+\varepsilon}(Q)}. \quad (11)$$

Recalling that $u\perp 1$ on Q, one has for any $z\in Q$

$$\int_Q|u(x)|^{1+\varepsilon}dx\leq\int_Q\int_Q|u(x)-u(y)|^{1+\varepsilon}dx\,dy\leq 2^{1+\varepsilon}\int_Q|u(x)-u(z)|^{1+\varepsilon}\,dx.$$

Hence and by the obvious inequality $\int_{2Q}\frac{dz}{|x-z|^{n-\varepsilon(1+\varepsilon)}}>\int_{|z-x|<1/2}\frac{dz}{|x-z|^{n-\varepsilon(1+\varepsilon)}}=\frac{|(1-\varepsilon)^{n-1}|}{\varepsilon(1+\varepsilon)2^{\varepsilon(1+\varepsilon)}}$,

where $x\in Q$, it follows that $\int_Q|u(x)|^{1+\varepsilon}dx\leq\frac{2^{(1+\varepsilon)^2}\varepsilon(1+\varepsilon)}{|(1-\varepsilon)^{n-1}|}\int_{2Q}\int_Q\frac{|u(x)-u(z)|^{1+\varepsilon}}{|x-z|^{n-\varepsilon(1+\varepsilon)}}dx\,dz.$

Thus, $\|u\|_{L^{1+\varepsilon}(Q)}\leq 2^{2+n/(1+\varepsilon)}n^{1/2}\left(\frac{\varepsilon(1+\varepsilon)}{|(1-\varepsilon)^{n-1}|}\right)^{1/(1+\varepsilon)}\|u\|_{\mathcal{W}_\perp^{1-\varepsilon,1+\varepsilon}(Q)}.$

Combining this inequality with (10), we justify (9) and hence complete the proof.

Now we give Sobolev embedding.

Theorem 3. Let $n\geq 1, 0\leq\varepsilon<1$, and $1-\varepsilon^2<n$. Then, for an arbitrary function $u\in\mathcal{W}_0^{1-\varepsilon,1+\varepsilon}(\mathbf{R}^n)$, there holds

$$\|u\|_{L^q(\mathbf{R}^n)}^{1+\varepsilon}\leq c(n,(1+\varepsilon))\frac{\varepsilon(1-\varepsilon)}{(n-(1-\varepsilon^2))^{\varepsilon}}\|u\|_{\mathcal{W}_0^{1-\varepsilon,1+\varepsilon}(\mathbf{R}^n)}^{1+\varepsilon}, \quad (12)$$

where $q=n(1+\varepsilon)/(n-(1-\varepsilon^2))$ and $c(n,(1+\varepsilon))$ is a function of n and $(1+\varepsilon)$.

From Theorem 1, one can derive inequality (1) for all $(1-\varepsilon)\in(0,1)$ with a constant c depending both on n and $(1+\varepsilon)$.

In the case $\varepsilon\leq 1/2$ considered in [3], one has $1<(1+\varepsilon)<2n$ and therefore the dependence of the constant c on $(1+\varepsilon)$ can be eliminated.

Thus, we arrive at the Bourgain–Brezis–Mironescu result and extend it to the values $\varepsilon\leq 1/2$.

The proof given in [3] relies upon some advanced harmonic analysis and is quite complicated. Our proof of (12) is straightforward and rather simple.

It is based upon an estimate of the best constant in a Hardy-type inequality for the norm in $\mathcal{W}_0^{1-\varepsilon,1+\varepsilon}(\mathrm{R}^n)$.

Proof: It is well known that the fractional Sobolev norm of order $(1-\varepsilon) \in (0,1)$ is non-increasing with respect to symmetric rearrangement of functions decaying to zero at infinity (see [4], [5], [6], [7]).

Let $v(|x|)$ denote the rearrangement of $|u(x)|$.

Then

$$\|u\|_{L^q(\mathbf{R}^n)} = \left(\frac{|(1-\varepsilon)^{n-1}|}{n}\int_0^\infty v(r)^q\, d(r^n)\right)^{1/q}, \qquad (13)$$

where $|(1-\varepsilon)^{n-1}|$ is the area of the unit sphere $(1-\varepsilon)^{n-1}$. Recalling that an arbitrary non-negative non-increasing function f on the semi-axis $(0,\infty)$ satisfies

$$\int_0^\infty f(t)^\lambda d(t^\lambda) \le \int_0^\infty \left(\int_0^t f(\tau)\, d\tau\right)^{\lambda-1} f(t)dt = \left(\int_0^\infty f(t)\, dt\right)^\lambda, \qquad \lambda \ge 1$$

the right-hand side in (13) does not exceed

$$\left(\frac{|(1-\varepsilon)^{n-1}|}{n}\right)^{1/q}\left(\int_0^\infty v(r)^{(1+\varepsilon)}\, d\left(r^{n-(1-\varepsilon^2)}\right)\right)^{1/(1+\varepsilon)}$$

$$= \frac{(n-(1-\varepsilon^2))^{1/(1+\varepsilon)}}{n^{1/q}|(1-\varepsilon)^{n-1}|^{(1-\varepsilon)/n}}\left(\int_{\mathbf{R}^n} v(|x|)^{(1+\varepsilon)}\frac{dx}{|x|^{(1-\varepsilon^2)}}\right)^{1/(1+\varepsilon)}.$$

We now see that (12) results from inequality (2).

Corollary 4. Let $n \ge 1, 0 \le \varepsilon < 1$, and $(1-\varepsilon^2) < n$.

Then any function $u \in \mathcal{W}_\perp^{1-\varepsilon,1+\varepsilon}(Q)$ satisfies

$$\|u\|_{L^{1+\varepsilon}(Q)}^{1+\varepsilon} \le c(n,(1+\varepsilon))\frac{\varepsilon}{(n-(1-\varepsilon^2))^\varepsilon}\|u\|_{\mathcal{W}_\perp^{1-\varepsilon,1+\varepsilon}(Q)}^{1+\varepsilon}.$$

Theorem 5. For any function $u \in \bigcup_{0<\varepsilon<1}\mathcal{W}_0^{1-\varepsilon,1+\varepsilon}(\mathrm{R}^n)$, there exists the limit

$$\lim_{\varepsilon\downarrow 1} s\|u\|_{\mathcal{W}_0^{1-\varepsilon,1+\varepsilon}(\mathbf{R}^n)}^{1+\varepsilon} = 2(1+\varepsilon)^{-1}|(1-\varepsilon)^{n-1}|\|u\|_{L^{1+\varepsilon}(\mathbf{R}^n)}^{1+\varepsilon}.$$

Proof. Since d can be chosen arbitrarily small, inequality (9) implies

$$\liminf_{\varepsilon\downarrow 1} s\|u\|^{1+\varepsilon}_{\mathcal{W}_0^{1-\varepsilon,1+\varepsilon}(\mathbf{R}^n)} \geq 2(1+\varepsilon)^{-1}|(1-\varepsilon)^{n-1}|\|u\|^{1+\varepsilon}_{L^{1+\varepsilon}(\mathbf{R}^n)}. \qquad (14)$$

Let us majorize the upper limit. By (14), it suffices to assume that $u \in L^{1+\varepsilon}(R^n)$. Clearly,

$$(1-\varepsilon)\|u\|^{1+\varepsilon}_{\mathcal{W}_0^{1-\varepsilon,1+\varepsilon}(\mathbf{R}^n)}$$

$$\leq 2\left\{\left((1-\varepsilon)\int_{\mathbf{R}^n}\int_{|y|\geq 2|x|}\frac{dy}{|x-y|^{n+(1-\varepsilon^2)}}|u(x)|^{1+\varepsilon}\,dx\right)^{1/(1+\varepsilon)}\right.$$

$$\left.+\left((1-\varepsilon)\int_{\mathbf{R}^n}|u(y)|^{1+\varepsilon}\int_{|y|\geq 2|x|}\frac{dx\,dy}{|x-y|^{n+(1-\varepsilon^2)}}\right)^{1/(1+\varepsilon)}\right\}^{(1+\varepsilon)}$$

$$+2(1-\varepsilon)\int_{\mathbf{R}^n}\int_{|x|<|y|<2|x|}\frac{|u(x)-u(y)|^{1+\varepsilon}}{|x-y|^{n+(1-\varepsilon^2)}}dx\,dy.$$

The first term in braces does not exceed

$$\left((1-\varepsilon)\int_{\mathbf{R}^n}\int_{|y|\geq|x|}\frac{dy}{|x-y|^{n+(1-\varepsilon^2)}}|u(x)|^{1+\varepsilon}\,dx\right)^{1/(1+\varepsilon)}$$

$$=\frac{|(1-\varepsilon)^{n-1}|^{1/(1+\varepsilon)}}{(1+\varepsilon)^{1/(1+\varepsilon)}}\left(\int_{\mathbf{R}^n}\frac{|u(x)|^{1+\varepsilon}}{|x|^{(1-\varepsilon^2)}}dx\right)^{1/(1+\varepsilon)}$$

hence its $\limsup_{\varepsilon\downarrow 1}$ is dominated by $|(1-\varepsilon)^{n-1}|^{1/(1+\varepsilon)}(1+\varepsilon)^{-1/(1+\varepsilon)}\|u\|_{L^{1+\varepsilon}(\mathbf{R}^n)}$.

The second term in braces is not greater than

$$(1-\varepsilon)^{1/(1+\varepsilon)}\left(2^{n+(1-\varepsilon^2)}\int_{\mathbf{R}^n}\frac{|u(y)|^{(1+\varepsilon)}}{|y|^{n+(1-\varepsilon^2)}}dy\int_{|x|<|y|/2}dx\right)^{1/(1+\varepsilon)}$$

$$=2^{(1-\varepsilon)}\left(\frac{(1-\varepsilon)}{(1+\varepsilon)}|(1-\varepsilon)^{n-1}|\right)^{1/(1+\varepsilon)}\left(\int_{\mathbf{R}^n}\frac{|u(y)|^{(1+\varepsilon)}}{|y|^{1-\varepsilon^2}}dy\right)^{1/(1+\varepsilon)},$$

so it tends to zero as $\varepsilon \downarrow 1$. We claim that

$$\limsup_{\varepsilon\downarrow 1}\int_{\mathbf{R}^n}\int_{|x|<|y|<2|x|}\frac{|u(x)-u(y)|^{(1+\varepsilon)}}{|x-y|^{n+(1-\varepsilon^2)}}dx\,dy = 0. \qquad (15)$$

By assumption of the theorem, $u \in \mathcal{W}_0^{\tau,1+\varepsilon}(R^n)$ for a certain $\tau \in (0,1)$. Let N be an arbitrary number greater than 1 and let $(1-\varepsilon) < \tau$. We have

$$2(1-\varepsilon)\int_{\mathbf{R}^n}\int_{|x|<|y|<2|x|}\frac{|u(x)-u(y)|^{1+\varepsilon}}{|x-y|^{n+(1-\varepsilon^2)}}dx\,dy$$

$$\leq 2(1-\varepsilon)N^{(1+\varepsilon)(\tau-(1-\varepsilon))} \int_{\mathbf{R}^n} \int_{\substack{|x|<|y|<2|x| \\ |x-y|\leq N}} \frac{|u(x)-u(y)|^{1+\varepsilon}}{|x-y|^{n+\tau(1+\varepsilon)}} dx\, dy$$

$$+ 2(1-\varepsilon) \int_{\mathbf{R}^n} \int_{\substack{|x|<|y|<2|x| \\ |x-y|>N}} \frac{|u(x)-u(y)|^{1+\varepsilon}}{|x-y|^{n+(1-\varepsilon^2)}} dx\, dy.$$

The first term in the right-hand side tends to zero as $\varepsilon \downarrow 1$ and the second one does not exceed

$$2^{\varepsilon+2}(1-\varepsilon) \int_{|x|>N/3} \int_{|x-y|>N} \frac{dy}{|x-y|^{n+(1-\varepsilon^2)}} |u(x)|^{1+\varepsilon} dx \leq c(n,(1+\varepsilon)) \int_{|x|>N/3} |u(x)|^{1+\varepsilon}\, dx,$$

which is arbitrarily small if N is sufficiently large. The proof is complete.

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