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QUADRATIC RANDOM INTEGRO- DIFFERENTIAL EQUATION

P.R.Shinde

Abstract

In this paper, we investigat abstract measure integro random differential equation and prove existence results random fixed point theorem of Dhage.

Keywords:

Random integro-differential equation;measure integral;random fixedpoint theorem; Caratheodory condition; banach algebra.

Author correspondence:

Department of Mathematics,
Gramin Mahavidhyalaya,
Vasat Nagar, Mukhed,Dist. Nanded (M.S.) INDIA

1.INTRODUCTION

Let X be the real Banach algebra with convenient norm $\|.\|$. Let x ,y ∈ X .Then the line segment $\overline{xy}$ in X is defined by

$$\overline{xy}=\{z\in X \mid z=x+r(y-x),0\leq r\leq 1\}$$

(1.1)

Let $x_o \in$ X be a fixed point and z ∈ X. Then for any $x \in \overline{x_0 z}$ We define the sets S_x and $\overline{S}_x$

$$\left.\begin{aligned} S_x &= \{rx|z = -\infty < r < 1\} \\ \overline{S}_x &= \{rx|z = -\infty < r \le 1\} \end{aligned}\right\}$$

(1.2)

Let $x_1, x_2 \in \overline{xy}$ be arbitrary .We say $x_1 < x_2$ if $S_{x_1} \subset S_{x_2}$ or ,equivalently $\overline{x_0 x_1} \subset \overline{x_0 x_2}$ .In this case we also write $x_2 > x_1$.

Let M denotes the σ-algebra of all subsets of X such that such that (X,M) is a measurable space .Let ca(X,M) be the space of all vector measures (real signed measures) and define a norm $\|.\|$ on ca(X,M) by

$$\|p\| = |p|(\mathrm{X})$$

(1.3)

Where $|p|$ is a total variation measure of p and is given by

$$|p|(\mathrm{X}) = \sup \sum_{i=1}^{\infty} |p(\mathrm{E}_i)|, E_i \subset X \ ,$$

(1.4)

Where supremum is taken over all possible partitions $\{E_i : i \in N\}$ of X. It is known that ca(X,M) is a Banach space with respect to the norm $\|.\|$,given by (1.3) .for any nonempty subset S of X, let $L^1_\mu(\mathrm{S},\mathrm{R})$ denote the space of μ –integrable real valued- functions on S which is equipped with the norm $\|.\|_{L^1_\mu}$ is given by

$$\|\phi\|_{L^1_\mu} = \int_S |\varphi(\mathrm{x})| \, d\mu.$$

For $\phi \in L^1_\mu(\mathrm{S},\mathrm{R})$. Let $p_1, p_2 \in ca(\mathrm{X},\mathrm{M})$ and define a multiplication composition o in $ca(\mathrm{X},\mathrm{M})$ by

$$(p_1 o p_2)(E) = p_1(\mathrm{E})\mathrm{p}_2(\mathrm{E})$$

For all $E \in M$ .Then we have .

Lemma 1.1. $ca(\mathrm{X},\mathrm{M})$ is a Banach algebra.

Let μ be a σ finite measure on X,and let $\mathrm{p} \in ca(\mathrm{X},\mathrm{M})$ .We say p is absolutely continuous with respect to the measure μ if μ(E) =0 for some E $\in$ M. In this case we also wright

$p << \mu$.

Let $x_0 \in X$ be a fixed and let M_0 denote the $\sigma-$algebra on S_{x_0} .Let $z \in X$ be such that $z > x_0$ and let M_z denotes the σ algebra of all sets containing M_0 and the sets of the form $\bar{S}_x, x \in \overline{x_0 z}$.

Given a $p \in ca(\mathrm{X},\mathrm{M})$ with $p << \mu$,consider the abstract measure integro random differential equation of the form

$$\frac{d}{d\mu}\left(\frac{p(\bar{S}_x)}{f\left(x, p(\bar{S}_x), \omega\right)}\right) = g\left(x, p(\bar{S}_x), \int_{\bar{S}_x} h\left(t, p(\bar{S}_t)\right) d\mu\right) \text{ a.e. } [\mu] \text{ on } \overline{x_0 z}$$

(1.5)

And $p(E) = q(E) \quad , E \in M_0$

(1.6)

Where q is a given known vector measure , $\omega \in X$, $\lambda\left(\bar{S}_x\right) = \dfrac{p\left(\bar{S}_x\right)}{f\left(x, p\left(\bar{S}_x\right), \omega\right)}$ is a signed measure such that $\lambda << \mu, \dfrac{d\lambda}{d\mu}$ is a Radon-Nikodym derivative of λ with respect to μ , $f : S_x \times R \times X \to R - \{0\}$, $g : S_z \times R \times R \to R$ and the map $x \to g\left(x, p(\bar{S}_x), \int_{\bar{S}_x} h\left(t, p(\bar{S}_t)\right) d\mu\right)$ is μ- integrable for each $p \in ca(X, M_z)$.

Definition 1.1. Given an initial real measure q on M_0 ,a vector $p \in ca(S_z, M_z)$ $(z > x_0)$ is said to be solution of problem (1.5)-(1.6),if

(i) $p(E) = q(E)$, $E \in M_0$

(ii) $p \ll \mu$ on $\overline{x_0 z}$,and

(Iii) p satisfies (5.2.5) a.e. $[\mu]$ on $\overline{x_0 z}$

Remark1.1. The problem (1.5)-(1.6),is equivalent to the abstract measure integral equation

$$P(E) = \left[f\left(x, p(E_1), \omega\right) \right] \int_E g\left(x, p(\bar{S}_x), \int_{\bar{S}_x} h\left(t, p(\bar{S}_t)\right) d\mu\right) d\mu \qquad E \in M_z, E \subset \overline{x_0 z}$$

(1.7)

and

$$p(E)=q(E) \text{ if } E\in M_0$$

(1.8)

A solution p of problem (5.2.5)-(5.2.6) in $\overline{x_0 z}$ will be denoted by $p(S_{x_0},q)$.

Note that our problem(1.5)-(1.6),includes the abstract measure differential equation considered in Dhage and Bellale [6] as a special case .To see this ,define $f(x,y,\omega)=1$ for all $x\in\overline{x_0 z}$ and $y\in R$ .then problem(5.2.5)-(5.2.6) reduces to

$$\frac{dp}{d\mu}=g\left(x,p\left(\overline{S_x}\right),\int_{\overline{S_x}} h\left(t,p\left(\overline{S_t}\right)\right)d\mu\right) \quad \text{a.e. } [\mu] \text{ on } \overline{x_0 z} \ ,$$

(1.9)

And $$p(E)=q(E)\ ,\quad E\in M_0$$

(1.10)

Thus ,problem (1.5)-(1.6),is more general.

2. AUXILIARY RESULTS

Let X be a Banach algebra and let $T:X\to X$ .T is called as compact if $\overline{T(X)}$ is a compact subset of X. T is called as totally bounded if for any bounded subset S of X , $T(S)$ is totally bounded subset of X . T is called as completely continuous if T is continuous and totally bounded on X .Every compact operator is totally bounded but converse may not be true ,however ,two notions are equivalent on a bonded subset of X .

An operator $T:X\to X$ is called D-Lipschitz if there exist a continuous and non decreasing function $\psi:R^+\to R^+$ such that

$$\|Tx-Ty\|\le\psi(\|x-y\|)$$

(2.1)

For all $x,y\in X$,where $\psi(0)=0$ .The function ψ is called $D-$function of T on X .In particular ,if $\psi(r)=\alpha r,\alpha>0$. T is called a Lipschitz with Lipschitz constant α .Further if $\alpha<1$,then T is called contraction with contraction constant α .Again if $\psi(r)<r$ for $r>0$,then T is called a nonlinear contraction on X with D -function ψ .

Now we are ready to prove the main result of this section.

Theorem2.1. (Dhage [4]).Let U and $\overline{U}$ denote respectively the open and closed bounded subset of a Banach algebra X such that $0 \in U$. Let $A, B : \overline{U} \to X$ be two operators such that,

a) A is D Lipschitz

b) B is completely continuous ,and

c) $M\phi(r) < r, r > 0$,where $M = \left\| B\left(\overline{U}\right) \right\|$,

then either

i) the equation $AxBx = x$ has a solution in $\overline{U}$ or

ii) there is a point $u \in \partial U$ such that $u = \lambda AuBu$ for some $0 < \lambda < 1$,where ∂U is a boundary of

U in X .

An interesting corollary to theorem 2.1 in the applicable form is

Corollary 2.1. Let $B_r(0)$ and $\overline{B}_r(0)$ denote respectively the open and closed balls in a Banach algebra centered at origin O of radius r for some real number $r > 0$ .Let $A, B : \overline{B_r}(0) \to X$ be two operators such that

$a)$ A is Lipschitz with Lipschitz constant α

$b)$ B is compact and continuous and

$c)$ $\alpha M < 1$,where $M = \left\| B\left(\overline{B}_r(0)\right) \right\|$.

Then either

(i) the operator equation $AxBx = x$ has a solution x in X with $\|x\| \le r$, or

(ii) there is an $u \in X$ with $\|u\| = r$ such that $\lambda AuBu = u$ for some $0 < \lambda < 1$

We define an order relation $\le$ in $ca(S_z, M_z)$ with the help of cone K in $ca(S_z, M_z)$ given by

$$K = \left\{ p \in ca(S_z, M_z) \mid p(E) \ge 0 \text{ for all } E \in M_z \right\}$$

(2.2)

Thus for any $p_1, p_2 \in ca(X, M)$ we have

$p_1 \le p_2$ if and only if $p_2 - p_1 \in K$

(2.3)

Or equivalently $p_1 \le p_2 \Leftrightarrow p_1(E) \le p_2(E)$ for all $E \in M_z$ (2.4)

Obviously the cone K is positive in $ca(S_z, M_z)$. To see this ,let $p_1, p_2 \in K$ .then $p_1(E) \ge 0$ and $p_2(E) \ge 0$ for all $E \in M_z$ .By multiplication composition

$$(p_1 o p_2)(E) = p_1(E) p_2(E) \ge 0$$

For all $E \in M_z$ .As a result $p_1 o p_2 \in K$, so K is positive cone in $ca(S_z, M_z)$.

The following lemmas follow immediately from the definition of positive cone K in $ca(S_z, M_z)$

Lemma 2.1(Dhage1).If $u_1, u_2, v_1, v_2 \in K$ are such that $u_1 \le v_1$ and $u_2 \le v_2$,then $u_1 u_2 \le v_1 v_2$.

Lemma 2.2. The cone K is normal in $ca(S_z, M_z)$.

Proof. To finish it is enough to prove that the norm $\|.\|$ is semi-monotone on K. Let $p_1, p_2 \in K$ be such that $p_1 \le p_2$ on M_z .Then we have

$$0 \le p_1(E) \le p_2(E)$$

For all $E \in M_z$.

Now for a countable partition $\sigma = \{E_n : n \in N\}$ of S_z ,one has

$$\begin{aligned} \|p\| &= |p_1|(S_z) \\ &= \sup_{\sigma} \sum_{i=1}^{\infty} |p_1(E_i)| \\ &\le \sup_{\sigma} \sum_{i=1}^{\infty} |p_2(E_i)| \\ &= |P_2|(S_z) \\ &= \|p_2\| \end{aligned}$$

As a result $\|.\|$ is semi-monotone on K and consequently the cone K is normal in $ca(S_z, M_z)$.

Proof of the lemma is complete .

An operator $T :\ X \to X$ is called positive if the range $r(T)$ of T is contained in the cone K in K .

Theorem 2.2(Dhage[4]). Let $[u,v]$ be an order interval in the real Banach algebra X and let $A,B:[u,v]\to[u,v]$ be positive and nondecreasing operators such that

(a) A is Lipschitz with a Lipschitz constant α ,

(b) B is compact and continuous ,and

(c) there exist elements $u,v\in X$ with $u\le v$ satisfy $u\le AuBu$ and $AvBv\le v.$

Further ,if the cone K is positive and normal ,then the operator equation $AxBx=x$ has a least and greatest positive solution in $[u,v]$,whenever $\alpha M<1$,where

$$M=\|B[u,v]\|=\sup\{\|Bx\|:x\in[u,v]\}$$

Theorem 2.3 (Dhage [4]). Let K be a positive cone in a real Banach algebra X and let $A,B:K\to K$ be nondecreasing operators such that

(a) A is Lipschitz with a Lipschitz constant α ,

(b) B is totally bounded and

(c) there exist elements $u,v\in K$ with $u\le v$ satisfy $u\le AuBu$ and $AvBv\le v.$

Further ,if the cone K is positive and normal ,then the operator equation $AxBx=x$ has a least and greatest positive solution in $[u,v]$,whenever $\alpha M<1$,where

$$M=\|B[u,v]\|=\sup\{\|Bx\|:x\in[u,v]\}$$

3.Existence Results

We need the following defination in the sequel .

Defination 3.1. A function $\beta:S_z\times R\times R\to R$ is called Caratheodory if

(i) $x\to\beta(x,y_1,y_2)$ is μ-measurable for each $y_1,y_2\in R$, and

(ii) $(y_1,y_2)\to\beta(x,y_1,y_2)$ is continuous almost everywhere $[\mu]$ on $\overline{x_0z}$.

A Carathe'odory function β on $S_z\times R\times R$ is called L^1_μ -Carathe'odory if

For each real number $r>0$ there exists a function $h_r\in L^1_\mu(S_z,R_+)$ such that

$$|\beta(x,y_1,y_2)|\le h_r(x) \text{ a.e. } [\mu] \text{ on } \overline{x_0z} .$$

For all $y_1,y_2\in R$ with $|y_1|\le r$ and $|y_2|\le r$.

A function $\psi:R_+\to R_+$ is called submultiplicative if $\psi(\lambda r)\le\lambda\psi(r)$ for all real number $\lambda>0$

Let Ψ denotes the class of functions $\psi : R_+ \to R_+$ satisfying following properties : ψ

(i) ψ is continuous ,

(ii) ψ is nondecreasing, and

(iii) ψ is submultiplicative.

A function $\psi \in \Psi$ is called a D-function on R_+ .There do exist D functions ,in facts ,the function $\psi : R_+ \to R_+$ defined by $\psi(\lambda) = \lambda r$, $\lambda > 0$ is a D-function on R_+ .

We consider the following set of assumptions :

(A_0) For any $z > x_0$,the σ- algebra M_z is compact with respect to the topology generated by the Pseudo-metric d defined on M_z by

$$D(E_1, E_2) = |\mu|(E_1 \Delta E_2) \ , E_1, E_2 \in M_z \ .$$

(A_1) The function $x \to |f(x, o, \omega)|$ is bounded with $F_0 = \sup_{x \in S_z} |f(x, 0, \omega)|$.

(A_2) The function f is continuous and there exists a bounded function functions $\alpha : S_z \to R^+$

With bound $\|\alpha\|$ such that

$$|f(x, y_1, \omega) - f(x, y_2, \omega)| \le \alpha(x, \omega)|y_1 - y_2| \quad \text{a.e.}[\mu] \ , \ x \in \overline{x_0 z}$$

For all $y_1, y_2 \in R$.

(B_0) q is continuous on M_z with respect to the Pseudo metric d defined in (A_0) .

(B_1) The function $x \to h(x, p(\bar{S}_x))$ is μ integrable and satisfies

$$|h(t, y)| \le \gamma(x)|y| \ \text{a.e. on } \overline{x_0 z} \quad \text{for all } y \in R$$

(B_2) The function $g(x, y_1, y_2)$ is Carathe'odory.

(B_3) There exists a function $\phi \in L^1_\mu(S_z, R^+)$ such that $\phi(x) > 0$ a.e. $[\mu]$ on $\overline{x_0 z}$ and a D-function $\psi : [0, \infty) \to (0, \infty)$ such that

$$|g(x, y_1, y_2)| \le \phi(x)\psi(|y_1| + |y_2|) \ \text{a.e.}[\mu] \text{ on } \overline{x_0 z} \quad \text{for all } y_1, y_2 \in R \ .$$

We frequently use the following estimate of the function g in the subsequent part of the paper. For any $p \in ca(S_z, M_z)$, one has

$$\left| g\left(x, p\left(\bar{S}_x\right), \int_{\bar{S}_x} h\left(t, p\left(\bar{S}_t\right)\right) d\mu \right) \right|$$

$$\le \phi(x)\psi\left(\left| p(S_x) \right| + \int_{\bar{S}_x} \left| k\left(t, p\left(\bar{S}_t\right)\right) d\mu \right| \right)$$

$$\le \phi(x)\psi\left(|p|(S_z) + \int_{\bar{S}_x} \left| \gamma(x)\left(p\left(\bar{S}_t\right)\right) \right| d\mu \right)$$

$$\le \phi(x)\psi\left(\|p\| + \int_{\bar{S}_x} \gamma(x)\|p\| d\mu \right)$$

$$\le \phi(x)\psi\left(\|p\| + \|\gamma\|_{L^1_\mu} \|p\| \right)$$

$$\le \phi(x)\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi\left(\|p\|\right)$$

Theorem 3.1. Suppose that the assumptions $(A_0)-(A_2)$ and $(B_0)-(B_3)$ holds .Suppose that there exist a real number $r > 0$ such that

$$r > \frac{F_0\left[\|q\| + \|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(r)\right]}{1-\|\alpha\|\left[\|q\| + \|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(r)\right]}$$

(3.1)

Where $\|\alpha\|\left[\|q\| + \|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(r)\right] < 1$ and $F_0 = \sup_{x \in S_z} \left| f(x, 0, \omega) \right|$ .Then the problem ((1.5)-(1.6), has a solution on $\overline{x_0 z}$.

Proof. Consider an open ball $\bar{B}_r(0)$ in $ca(S_z, M_z)$ centered at the origin and radius r ,where r satisfies the inqualities in (3.1). Define two operators

$$A, B : \bar{B}_r(0) \to \mathrm{ca}(S_z, M_z) \text{ by}$$

$$A_p(E) = 1 \qquad \text{if } E \in M_0 \tag{3.2}$$

$$A_p(E) = f(x, p(E), \omega) \qquad \text{if } E \in M_z, E \subset \overline{x_0 z}$$

and

$$B_p(E) = q(E) \qquad \text{if} \qquad E \in M_0 \tag{3.3}$$

$$B_p(E)=\int_E g\left(x,p(\bar{S}_x),\int_{\bar{S}_x} h(t,p(\bar{S}_t))d\mu\right)d\mu \qquad \text{if } E\in M_z\ ,E\subset \overline{x_0 z}$$

We shall show that the operators A and B satisfies all the condition of corollary2.1 on $\bar{B}_r(0)$.

Step I : First, we show that A is Lipschitz on $\bar{B}_r(0)$ .Let $p_1,p_2\in \bar{B}_r(0)$ be arbitrary .Then by assumption $(A_{2)}$,

$$\left|A_{p_1}(E)-A_{p_2}(E)\right|=\left|f(x,p_1(E),\omega)-f(x,p_2(E),\omega)\right|$$

$$\le \alpha(\mathrm{x},\omega)\left|p_1(E)-p_2(E)\right|$$

$$\le \|\alpha\|\,|p_1-p_2|(E)$$

For all $E\in M_z$ .Hence by definition of the norm in $ca(S_z,M_z)$ one has

$$\left\|A_{p_1}-A_{p_2}\right\|\le\|\alpha\|\|p_1-p_2\|$$

For all $p_1,p_2\in ca(S_z,M_z)$ .As a result A is a Lipschitz operator on $\bar{B}_r(0)$ with the Lipschitz constant $\|\alpha\|$.

Step II : We show that B is a continuous on $\bar{B}_r(0)$ .Let $\{p_n\}$ be a sequence of vector measures in $\bar{B}_r(0)$ converging to a vector measure p .Then by dominated convergence theorem ,

$$\lim_{n\to\infty}\bar{B}_{p_n}(E)=\lim_{n\to\infty}\int_E g\left(x,p_n(\bar{S}_x),\int_{\bar{S}_x} h(t,p_n(S_t))d\mu\right)d\mu$$

$$=\int_E g\left(x,p(\bar{S}_x),\int_{\bar{S}_x} h(t,p(\bar{S}_t))d\mu\right)d\mu$$

$$=\bar{B}_p(E)$$

For all $E\in M_z, E\subset\overline{x_o z}$ .Simillarly if $E\in M_0$,then

$$\lim_{n\to\infty}\bar{B}_{p_n}(E)=q(E)=B_p(E)\ .$$

And so B is a continuous operator on $\bar{B}_r(0)$.

Step III : Next ,we show that B is a totally bounded operator on $\bar{B}_r(0)$. Let $\{p_n\}$ be a sequence in $\bar{B}_r(0)$ .Then we have $\|p_n\|\le r$ for all $n\in N$ .We shall show that the set

$\{B_{p_n} : n \in N\}$ is uniformly bounded and equi-continuous set in $ca(S_z, M_z)$. In this step, we first show that $\{B_{p_n}\}$ is uniformly bounded. Then there exists two subsets $F \in M_0$ and $G \in M_z$, $G \subset \overline{x_0 z}$,such that bounded.

Let $E \in M_z$.

$$E = F \cup G \text{ and } F \cap G = \phi .$$

Hence by definition of B ,

$$\left|B_{p_n}(E)\right| \le \left|q(F)\right| + \int_G \left| g\left(x, p_n(\bar{S}_x), \int_{\bar{S}_x} h(t, p_n(\bar{S}_t)) d\mu \right) \right| d\mu$$

$$\le \|q\| + \int_G \phi(x)\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi(\|p_n\|) d\mu$$

$$\le \|q\| + \int_E \phi(x)\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi(\|p_n\|) d\mu$$

$$\le \|q\| + \|\phi\|_{L^1_\mu}\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi(\|p_n\|) \quad \text{for all}$$

$E \in M_z$.

From (3.3) it follows that

$$\|B_{p_n}\| = |B_{p_n}|(S_z)$$

$$= \sup_\sigma \sum_{i=n}^{\infty} \left|B_{p_n}(E_i)\right|$$

$$= \|q\| + \|\phi\|_{L^1_\mu}\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi(\|p\|)$$

$$= \|q\| + \|\phi\|_{L^1_\mu}\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi(r)$$

For all $n \in N$ .Hence the sequence $\{B_{p_n}\}$ is uniformly bounded in $\bar{B}_r(0)$

Step IV: Next we show that $\{B_{p_n} : n \in N\}$ is a equi-continuous set in $ca(S_z, M_z)$ .Let $E_1, E_2 \in M_z$.

Then there exist subsets $F_1, F_2 \in M_0$ and $G_1, G_2 \in M_z$, $G_1 \subset \overline{x_0 z}$, $G_2 \subset \overline{x_0 z}$ such that

$$E_1 = F_1 \cup G_1 \text{ with } F_1 \cap G_1 = \phi$$

and

$$E_2 = F_2 \cup G_2 \text{ with } F_2 \cap G_2 = \phi$$

We know the identities

$$G_1 = (G_1 - G_2) \cup (G_2 \cap G_1)$$

(3.4)

$$G_2 = (G_2 - G_1) \cup (G_1 \cap G_2)$$

Therefore we have

$$B_{p_n}(E_1) - B_{p_n}(E_2) \leq q(F_1) - q(F_2) + \int_{G_1 - G_2} \left| g\left(x, p_n(\bar{S}_x), \int_{\bar{S}_x} h(t, p_n(\bar{s}_t)) d\mu \right) \right| d\mu$$

$$+ \int_{G_2 - G_1} g\left(x, p_n(\bar{S}_x), \int_{\bar{S}_x} h(t, p_n(\bar{S}_t)) d\mu \right) d\mu$$

Since g is Caratheodory and satisfies (B_3) , we have that

$$\left| B_{p_n}(E_1) - B_{p_n}(E_2) \right| \leq \left| q(F_1) - q(F_2) \right| + \int_{G_1 \Delta G_2} \left| g\left(x, p_n(\bar{S}_x), \int_{\bar{S}_x} h(t, p_n(\bar{s}_t)) d\mu \right) \right| d\mu$$

$$\leq \left| q(F_1) - q(F_2) \right| + \int_{G_1 \Delta G_2} \phi(x) \left(1 + \|\gamma\|_{L^1_\mu} \right) \psi(\|p_n\|) d\mu$$

Assume that

$d(E_1, E_2) = |\mu|(E_1 \Delta E_2) \to 0$.

Then we have $E_1 \to E_2$. As a result $F_1 \to F_2$ and $|\mu|(G_1 \Delta G_2) \to 0$ .As q is continuous on a compact M_z , it is uniformly continuous and so

$$\left| B_{p_n}(E_1) - B_{p_n}(E_2) \right| \leq \left| q(F_1) - q(F_2) \right| + \int_{G_1 \Delta G_2} \phi(x) \left(1 + \|\gamma\|_{L^1_\mu} \right) \psi(\|p_n\|) d\mu$$

$$\to 0 \text{ as } E_1 \to E_2$$

This shows that $\{B_{p_n} : n \in N\}$ is a equi-continuous set in $ca(S_z, M_z)$. Now an application of the Arzela-Ascolli theorem yields that B is a totally bounded operator on $\bar{B}_r(o)$ .Now B is continuous and totally bounded operator on $\bar{B}_r(o)$,it is completely continuous operator on $\bar{B}_r(o)$.

Step V: Finally we show that the hypothesis (c) of corollary 2.1 .The Lipschitz constant of A is $\|\alpha\|$ .Here the number M in the hypothesis (c) is given by

$$M = \left\| B\left(\bar{B}_r(0) \right) \right\|$$

$$=\sup\left\{\left\|B_p\right\|: p\in \bar{B}_r(\mathrm{r})\right\}$$

$$\sup\left\{\left|B_p\right|\left(S_z\right): p\in \bar{B}_r\left(0\right)\right\}$$

Now let $E\in M_z$ .Then there are sets $F\in M_0$ and $G\in M_z$,$G\subset \overline{x_0 z}$ such that

$$E=F\cup G \text{ and } F\cap G=\phi \ .$$

From the definition of of B it follows that

$$B_p\left(E\right)=q\left(F\right)+\int_G g\left(x, p\left(\bar{S}_x\right), \int_{\bar{S}_x} h\left(t, p\left(\bar{S}_t\right)\right) d\mu\right) d\mu$$

$$\left|B_p\left(F\right)\right|\le\left|q\left(F\right)\right|+\int_G\left|g\left(x, p\left(\bar{S}_x\right), \int_{\bar{S}_x} h\left(t, p\left(\bar{S}_t\right)\right) d\mu\right)\right| d\mu$$

$$\le\|q\|+\int_G \phi(\mathrm{x})\left(1+\|\gamma\|_{L^1_\mu}\right)\psi\left(\|p\|\right) d\mu$$

$$\le\|q\|+\int_{\overline{x_0 z}} \phi(\mathrm{x})\left(1+\|\gamma\|_{L^1_\mu}\right)\psi\left(\|p\|\right) d\mu$$

$$=\|q\|+\|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi\left(\|p\|\right) \ .$$

Hence ,from (4.4.6) it follows that

$$\left\|B_p\right\|\le\|q\|+\|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi\left(\|p\|\right)$$

For all $p\in \bar{B}_r(0)$ .As a results we have

$$M=\left\|B\left(\bar{B}_r(0)\right)\right\|\le\|q\|+\|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi\left(\|p\|\right)$$

Now

$$\alpha M\le\|\alpha\|\left[\|q\|+\|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi\left(r\right)\right]<1$$

And so hypothesis (c) of corollary2.1 yields that either the operator $AxBx=x$ has a solution , or there is a $u\in ca\left(S_z, M_z\right)$ such that $\|u\|=r$ satisfying $u=\lambda AxBx$ for some $0<\lambda<1$ .We show that this later assertion does not hold. Assume the contrary .Then we have

$$u\left(E\right)=\lambda\left[f\left(x, u\left(G\right),\omega\right)\right]\left(\int_E g\left(x, \mathrm{u}\left(\bar{S}_x\right), \int_{\bar{S}_x} h\left(t, \mathrm{u}\left(\bar{S}_t\right)\right) d\mu\right) d\mu\right) \quad \text{, if} \quad E\in M_z \quad ,\mathrm{E}\subset \overline{x_0 z}$$

,

$$=\lambda q(E)\ ,\text{ if } E\in M_0$$

For some $0<\lambda<1$.

If $E\in M_z$,then the sets $F\in M_0$ and $G\in \bar{M}_z$, $G\subset \overline{x_0 z}$ such that $E=F\cup G$ and $F\cap G=\phi$ Then we have

$$u(E)=\lambda Au(E)Bu(E)$$

$$=\lambda\left[f\left(x,u(G),\omega\right)\right]\left(q(F)\right)+\int_G g\left(x,\mathrm{u}\left(\bar{S}_x\right),\int_{\bar{S}_x} h\left(t,\mathrm{u}\left(\bar{S}_t\right)\right)d\mu\right)d\mu$$

$$=\lambda\left[f\left(x,u(G),\omega\right)-f\left(x,0,\omega\right)\right]\left(q(F)+\int_G g\left(x,\mathrm{u}\left(\bar{S}_x\right),\int_{\bar{S}_x} h\left(t,\mathrm{u}\left(\bar{S}_t\right)\right)d\mu\right)d\mu\right)$$

$$+\lambda\left[f\left(x,0,\omega\right)\right]\left(q(F)+\int_G g\left(x,\mathrm{u}\left(\bar{S}_x\right),\int_{\bar{S}_x} h\left(t,\mathrm{u}\left(\bar{S}_t\right)\right)d\mu\right)d\mu\right)$$

Hence

$$\left|u(E)\right|\le\lambda\left|f\left(x,u(G),\omega\right)-f\left(x,0,\omega\right)\right|\left(\left|q(F)\right|+\int_G\left|g\left(x,\mathrm{u}\left(\bar{S}_x\right),\int_{\bar{S}_x} h\left(t,\mathrm{u}\left(\bar{S}_t\right)\right)d\mu\right)\right|d\mu\right)$$

$$+\left|f\left(x,0,\omega\right)\right|\left(\left|q(F)\right|+\int_G\left|g\left(x,\mathrm{u}\left(\bar{S}_x\right),\int_{\bar{S}_x} h\left(t,\mathrm{u}\left(\bar{S}_t\right)\right)d\mu\right)\right|d\mu\right)$$

$$\le\lambda\left(\alpha(x)\left|u(G)\right|+F_0\right)\left(\|q\|+\int_G\phi(\mathrm{x})\left(1+\|\gamma\|_{L^1_\mu}\right)\psi\left(\|u\|\right)d\mu\right)$$

$$\le\left[\|\alpha\|\left|u\right|(E)+F_0\right]\left(\|q\|+\int_{\overline{x_0 z}}\phi(\mathrm{x})\left(1+\|\gamma\|_{L^1_\mu}\right)\psi\left(\|u\|\right)d\mu\right)$$

$$\le\left[\|\alpha\|\|u\|+F_0\right]\left[\|q\|+\|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi\left(\|u\|\right)\right]$$

(1.1) Which further implies that

$$\|u\|\le\left(\|\alpha\|\|u\|\left[\|q\|+\|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi\left(\|u\|\right)\right]\right)$$

$$+F_0\left[\|q\|+\|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi\left(\|u\|\right)\right]$$

$$\le\frac{F_0\left[\|q\|+\|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi\left(\|u\|\right)\right]}{1-\|\alpha\|\left[\|q\|+\|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi\left(\|u\|\right)\right]}$$

Substituting $\|u\| = r$ in the above inequality yields

$$r \le \frac{F_0\left[\|q\| + \|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(\mathrm{r})\right]}{1-\left(\|\alpha\|\left[\|q\| + \|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(\mathrm{r})\right]\right)}$$

(3.6)

Which is a contradiction to the first inequality (3.1) .In consequence ,the operator equation $p(E) = A_p(E)B_p(E)$ has a solution $u\left(\bar{S}_{x_o}, q\right)$ in $ca(S_z, M_z)$ with $\|u\| \le r$. This further implies that the problem (1.5)-(1.6),has a random solution on $\overline{x_0 z}$ .This complete the proof .

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