International Journal of Engineering, Science and Mathematics
Vol. 9 Issue 8, August 2020,
ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijesm.co.in, Email: ijesmj@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gage as well as in Cabell's Directories of Publishing Opportunities, U.S.A

EIGEN VALUES ESTIMATES ON RIEMANNIAN MANIFOLDS

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ABSTRACT

In this article we presented a concept of an eigenvalues on Riemannian manifolds, we estimated a value of aconstant $c_m[(g)]$ that Korevaar [3], used it to estimated a value of $\lambda_k(M,g)$. We proved theorems 4 and 5 by theorem 2.1 of [1] and we applied this theorem to the Steklov eigenvalue problem.

KEYWORDS:

Riemannian manifold, conformal class, Hölder inequality, Steklovproblem, eigenvalue.

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INTRODUCTION:

Let $(\boldsymbol{M},\boldsymbol{g})$ Is a compact orientable $\boldsymbol{m}$-dimensional Riemannian manifold. the spectrum of the Laplace operator acting on functionsis discrete and consists of a nondecreasing sequence $\{\boldsymbol{\lambda_k}(\boldsymbol{M},\boldsymbol{g})\}_{\boldsymbol{k=1}}^{\infty}$ of eigenvalues each occurring with finite multiplicity. By Weyl's law, the asymptotic behavior of $\boldsymbol{\lambda_k}$ is given by (see [2]).

$$\lambda_k(M,g) \sim \alpha_m \left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}}, k \longrightarrow \infty$$

where μ_g is the Riemannian measure associated with g, $\alpha_m = 4\pi^2 w_m^{-\frac{2}{m}}$ and w_m is the volume of the unit ball in the standard $\mathbb{R}^m$.

Korevaar [3], obtained the following upper bounds:

(i) If (M^m, g) is a compact Riemannian manifold of dimension m, then for every $k \in N^*$,

$$\lambda_k(M,g)\mu_g(M)^{\frac{2}{m}} \leq c_m([g])k^{\frac{2}{m}}, \qquad (1)$$

where $c_m([g])$ is a constant depending only on the conformal class $[g]$ of the metric g.

(ii) If (Σ_γ, g) is a compact orientable surface of genus γ, then for every $k \in N^*$,

$$\lambda_k(\Sigma_\gamma, g)\mu_g(\Sigma_\gamma) \leq C(\gamma+1)k, \qquad (2)$$

where C is a universal constant.

N. Korevaar [3], proved that If a compact Riemannian manifold (M, g) of dimension $m \geq 2$ is conformally equivalent to a Riemannian manifold with nonnegative Ricci curvature

$$\lambda_k(M,g)\mu_g(M)^{\frac{2}{m}} \leq B_m k^{\frac{2}{m}}, \qquad (3)$$

where B_m is a constant depending only on m.

Corollary .1 we show that for each integer $m \geq 2$

(i) $c_m[(g)] \geq B_m$

(ii) $Vol\left(\frac{B_m}{C_m}\right)^{2/m} \geq 0$

Proof. (i) when $m = 2$ Korevaar obtain the following bound in inequality (1)

$$\lambda_k(M,g)\mu_g(M)^{\frac{2}{m}} \leq c_m([g])k^{\frac{2}{m}},$$

With (3)

$$\lambda_k(M,g)\mu_g(M)^{\frac{2}{m}} \leq B_m k^{\frac{2}{m}},$$

Hence we have $C_m[(g)] \geq B_m$

Where $C_m[(g)]$ is a constant depending on the conformal class $[g]$ of metric .

(ii) Theorem .5 with (3) give that

$$A_m Vol([g])^{2/m} + B_m K^{2/m} \geq C_m([g])K^{2/m}$$
$$A_m Vol([g]) + C_m([g]) \geq C_m([g])$$

For (i) .By which we get

$$A_m Vol([g]) \geq 0\,,$$
$$Vol([g]) \geq 0\ .$$

Now we apply Theorem 2.1, of [1] to a special case of $m - m$ spaces which are Riemannian manifolds, in order to prove Theorem .5, and Theorem .4 . The arguments we use to prove these two theorems are similar. We start by giving in details the proof of Theorem .4.

Definition .2 let (M^m, g) be a Riemannian manifold of dimension m.The capacity of a capacitor (F, G) in M is defined by.

$$cap_g(F,G) = \inf_{\varphi \in \mathcal{T}} \int_M |\nabla_g \varphi|^2 \, d\mu_g,$$

Where $\mathcal{T} = \mathcal{T}(F, G)$ is the set of all compactly supported Lipschitz functions on M such that supp $\varphi \subset G^\circ = G \setminus \partial G$ and $\varphi \equiv 1$ in a neighborhood of F. If $\mathcal{T}(F, G)$ is empty, then $cap_g(F, G) = +\infty$. Similarly, we can define the m-capacity as.

$$cap_{[g]}^{(m)}(F,G) = \inf_{\varphi \in \mathcal{T}} \int_M |\nabla_g \varphi|^m \, d\mu_g,$$

Since m is the dimension of M, it is clear that the m-capacity depends onlyon the conformal class $[g]$ of the metric g.

Proposition .3 Under the assumptions of Theorem .4, take the $m-m$ space(Ω, d_{g0}, μ), where d_{g0} is the Riemannian distance corresponding to themetric g_0 and μ is a non-atomic finite measure on Ω. Then for every $n \in \mathbb{N}^*$, there exists a family of capacitors $\mathcal{A} = \{(F_i, G_i)\}_{i=1}^n$ with the following properties:

(i) $\mu(F_i) \geq \frac{\mu(\Omega)}{8c_m^2 n}$;

(ii) The G_i's are mutually disjoint;

(iii) The family $\mathcal{A}$ is such that either.

(a) All the F_i's are annuli, $G_i = 2F_i$ and $cap_{[g]}^{(m)}(F_i, 2F_i) \leq Q_m$, or

(b) All the F_i's are domains in Ω and $G_i = F_i^{r_0}$,

Where $r_0 = \frac{1}{1600}$ and, c_m and Q_m are constants depending only on the dimension,

Proof. Let us start with the observation that the metric space (Ω, d_{g_0}) satisfies the $(2, N; 1)$-covering property. For each ball $B(x, r)$ with center in Ω and radius smaller than 1, take a maximal family$\{B(x_i, r/4)\}$ of disjoint balls with centers in $B(x, r)$. Let k be the cardinal of that family. The family of balls $\{B(x_i, r/2)\}$ covers $B(x, r)$. Hence.

$$k \min_i \mu_{g_0}\left(B(x_i, r/4)\right) \leq \sum_i \mu_{g_0}\left(B(x_i, r/4)\right) \leq \mu_{g_0}\big(B(x, r + r/4)\big).$$

Take x_{i_0} such that $\mu_{g_0}\left(B\left(x_{i_0}, r/4\right)\right) = \min_i \mu_{g_0}\left(B(x_i, r/4)\right)$. We have

$$k \leq \frac{\mu_{g_0}\big(B(x, r + r/4)\big)}{\min_i \mu_{g_0}\left(B(x_i, r/4)\right)} \leq \frac{\mu_{g_0}\big(B(x, 2r)\big)}{\mu_{g_0}\left(B\left(x_{i_0}, r/4\right)\right)} \leq \frac{\mu_{g_0}\left(B\left(x_{i_0}, 4r\right)\right)}{\mu_{g_0}\left(B\left(x_{i_0}, r/4\right)\right)}.$$

Since $Ricci_{g_0}(\Omega) \geq -(m-1)$, we have $\forall 0 < s < r$,

$$\frac{\mu_{g_0}\big(B(x, r)\big)}{\mu_{g_0}\big(B(x, s)\big)} \leq \frac{\int_0^r \sinh^{m-1} t dt}{\int_0^s \sinh^{m-1} t dt}.$$

Since for every positive t one has $t \leq \sinh t \leq te^t$, we get

$$\frac{\mu_{g_0}\big(B(x, r)\big)}{\mu_{g_0}\big(B(x, s)\big)} \leq \left(\frac{r}{s}\right)^m e^{(m-1)r}.$$

In particular, we have

$$\mu_{g_0}\big(B(x,r)\big) \le r^m e^{(m-1)r} (4)$$

And, $\forall r < 1$,

$$k \le \frac{\mu_{g_0}\left(B\left(x_{i_0}, 4r\right)\right)}{\mu_{g_0}\left(B\left(x_{i_0}, r/4\right)\right)} \le 2^{4m} e^{4(m-1)r} =: C(r) \le C(1). \quad (5)$$

One can take $N = C(1)$ and deduce that (Ω, d_{g_0}) has the $(2, N; 1)$ covering property where N depends only on the dimension.

Now the proof of Proposition.3, is a straightforward consequence of Theorem2.1 of [1] recall that in the statement of Theorem 2.1 of [1], the constant c depends only on N. Therefore, in our case c depends only on the dimension. It remains to verify that in the case of annuli, there exists a constant Q_m depending only on the dimension such that for each i, we have

$$cap_{[g_0]}^{(m)}(F_i, 2F_i) \le Q_m$$

According to Theorem 2.1 of [1], the outer radii of the annuli we consider are smaller than one. It is enough to show that for each point $x \in \Omega$ and$0 \le r < R \le 1/2$, we have

$$cap_{[g_0]}^{(m)}(A, 2A) \le Q_m (6)$$

Where $A = A(x, r, R)$. Set

$$f(x) = \begin{cases} 1 & \text{if } x \in A(x,r,R) \\ \dfrac{2d_{g_0}\big(x, B(x, r/2)\big)}{r} & \text{if } x \in A(x, r/2, r) = B(x,r) \setminus B(x, r/2) \\ 1 - \dfrac{d_{g_0}\big(x, B(x,R)\big)}{R} & \text{if } x \in A(x,R,2R) = B(x,2B) \setminus B(x,R) \\ 0 & \text{if } x \in M \setminus A(x, r/2, 2R) \end{cases}$$

It is clear that $f \in \mathcal{T}(A, 2A)$ and

$$\left|\nabla_{g_0} f\right| \le \frac{2}{r}, \text{on } B(x,r) \setminus B(x, r/2)$$

$$\left|\nabla_{g_0} f\right| \le \frac{1}{R}, x \in B(x, 2R) \setminus B(x,R).$$

Therefore

$$cap_{[g_0]}^{(m)}(A, 2A) \le \int_M \left|\nabla_{g_0} f\right|^m d\mu_{g_0} \le \left(\frac{2}{r}\right)^m \mu_{g_0}\big(A(x, r/2, r)\big) + \left(\frac{1}{R}\right)^m \mu_{g_0}\big(A(x, R, 2R)\big).$$
$$\le \left(\frac{2}{r}\right)^m \mu_{g_0}\big(B(x,r)\big) + \left(\frac{1}{R}\right)^m \mu_{g_0}\big(B(x, 2R)\big).$$

Now since $r, 2R \in (0,1]$, Using inequality (4), one can control the last inequality by a constant Q_m depending only on the dimension which completes the proof of inequality (6).

Now we show how Theorem.5, follows from Proposition.3.

Theorem .4 Let (M, g_0) be a complete Riemannian manifold of dimension $m \ge 2$ with $Ricci_{g_0}(M) \ge -(m-1). Let \Omega \subset M$ be a relatively compact domain with C^1 boundary and g be any metric conformal to g_0.Then for every$k \in \mathbb{N}^*$, we have,

$$\lambda_k(\Omega, g)\mu_g(\Omega)^{\frac{2}{m}} \leq A_m^{'}\mu_{g_0}(\Omega)^{\frac{2}{m}} + B_m^{'}k^{\frac{2}{m}}, \qquad (7)$$

where $A_m^{'}$ and $B_m^{'}$ are constants depending only on the dimension m.

It is easy to see that we can derive from Theorem.5 and Theorem.6, inequalities $\lambda_k(M, g) \leq \frac{(m-1)^2}{4}a^2 + \beta_m\left(\frac{k}{\mu_g(M)}\right)^{2/m}$, as obtained by [4] and [5] but with different constants.

Proof. Take the $m - m$space$(\Omega, d_{g_0}, \mu_\Omega)$, where$\mu_\Omega = \mu_g\big|_\Omega$.According to Proposition.3, there exists a family$\{(F_i, G_i)\}$of $3k$ capacitors which satisfies the properties (i), (ii) and either(iii)(a) or (iii)(b) of the proposition. We consider each case separately.

Case 1. If$\{(F_i, G_i)\}_{i=1}^{3k}$is a family with the properties (i), (ii) and (iii)$(a)$of Proposition.3, then.

$$\lambda_k(\Omega, g) \leq A_m^{'}\left(\frac{k}{\mu_g(\Omega)}\right)^{\frac{2}{m}}, \qquad (8)$$

Where $A_m^{'} = 24c_m^2(2Q_m)^{\frac{2}{m}}$.

Indeed, we begin by choosing a family of $3k$ test functions $\{f_i : f_i \in \mathcal{T}(F_i, G_i)\}_{i=1}^{3k}$ such that

$$\int_M \left|\nabla_{g_0} f_i\right|^m d\mu_{g_0} \leq cap_{[g_0]}^{(m)}(F_i, G_i) + \epsilon.$$

Therefore,

$$R(f_i) = \frac{\int_\Omega \left|\nabla_g f_i\right|^2 d\mu_g}{\int_\Omega |f_i|^2 d\mu_g} \leq \frac{\left(\int_\Omega \left|\nabla_{g_0} f_i\right|^m d\mu_{g_0}\right)^{\frac{2}{m}}\left(\int_\Omega 1_{supp\, f_i} d\mu_g\right)^{1-\frac{2}{m}}}{\int_\Omega |f_i|^2 d\mu_g}$$
$$\leq \frac{\left(cap_{[g_0]}^{(m)}(F_i, G_i) + \epsilon\right)^{\frac{2}{m}}\left(\mu_\Omega(G_i)\right)^{1-\frac{2}{m}}}{\mu_\Omega(F_i)}. \qquad (9)$$

The first inequality follows from Hölder inequality and, because of the conformal invariance of$\int\left|\nabla_g f_i\right|^m d\mu_g$, we have replaced g by g_0. Since the G_i's are disjoint domains and $\sum_{i=1}^{3k}\mu_\Omega(G_i) \leq \mu_g(\Omega)$, at least k of them have measure smaller than $\frac{\mu_g(\Omega)}{k}$. Up to re-ordering, we assume that for the first k of the G_i's we have.

$$\mu_\Omega(G_i) \leq \frac{\mu_g(\Omega)}{k}. \qquad (10)$$

Now, we can take$\epsilon = \mathrm{Q_m}$. Using Proposition.3, (i) and (iii)(a) and inequality (48), we get from inequality (9).

$$R(f_i) \leq A_m^{'}\frac{\left(\frac{\mu_g(\Omega)}{k}\right)^{1-\frac{2}{m}}}{\frac{\mu_g(\Omega)}{k}} = A_m^{'}\left(\frac{k}{\mu_g(\Omega)}\right)^{\frac{2}{m}},$$

With $A_m^{'} = 24c_m^2(2Q_m)^{\frac{2}{m}}$, which completes the proof of Case 1.

Case 2. If $\{(F_i, G_i)\}_{i=1}^{3k}$ is a family with the properties (i), (ii) and $(iii)(b)$ of Proposition.3, then.

$$\lambda_k(\Omega, g) \le B_m' \left(\frac{\mu_{g_0}(\Omega)}{\mu_g(\Omega)}\right)^{\frac{2}{m}}, \qquad (11)$$

Where $B_m' = \frac{24c_m^2}{r_0^2}$.

Indeed, we define the test function f_i as follows.

$$f_i(x) = \begin{cases} 1 & \text{if} \quad x \in \quad F_i \\ 1 - \frac{d_{g_0}(x, F_i)}{r_0} & \text{if} \quad x \in \quad (G_i \setminus F_i) \\ 0 & \text{if} \quad x \in \quad G_i^c \end{cases}$$

We have $|\nabla_{g_0} f_i| \le \frac{1}{r_0}$. Therefore,

$$R(f_i) = \frac{\int_\Omega |\nabla_g f_i|^2 d\mu_g}{\int_M |f_i|^2 d\mu_g} \le \frac{\left(\int_\Omega |\nabla_{g_0} f_i|^m d\mu_{g_0}\right)^{\frac{2}{m}} \left(\int_\Omega 1_{supp\, f_i} d\mu_g\right)^{1-\frac{2}{m}}}{\int_\Omega |f_i|^2 d\mu_g} \le \frac{\frac{1}{r_0^2}\left(\mu_{g_0}(G_i \cap \Omega)\right)^{\frac{2}{m}} \left(\mu_{g_0}(G_i)\right)^{1-\frac{2}{m}}}{\mu_\Omega(F_i)} (12)$$

Since the G_i's are disjoint, we have.

$$\sum_{i=1}^{3k} \mu_{g_0}(G_i \cap \Omega) \le \mu_{g_0}(\Omega) and \sum_{i=1}^{3k} \mu_\Omega(G_i) \le \mu_g(\Omega).$$

Hence, there exist at least $2k$ sets among $G_1, \dots, G_{3k}$ such $\mu_{g_0}(G_i) \le \frac{\mu_{g_0}(\Omega)}{k}$. Similarly, there exist at least $2k$ sets (not necessarily the same ones) such that $\mu_g(G_i) \le \frac{\mu_g(\Omega)}{k}$. Therefore, up to re-ordering, we assume that the first k of G_i's satisfy both of the two following inequalities.

$$\mu_\Omega(G_i) \le \frac{\mu_g(\Omega)}{k} and \mu_{g_0}(G_i \cap \Omega) \le \frac{\mu_{g_0}(\Omega)}{k}. \qquad (13)$$

Using Proposition.3, (i) and inequalities (13), we get from inequality (12)

$$R(f_i) \le B_m' \frac{\left(\frac{\mu_{g_0}(\Omega)}{k}\right)^{\frac{2}{m}} \left(\frac{\mu_g(\Omega)}{k}\right)^{1-\frac{2}{m}}}{\frac{\mu_g(\Omega)}{k}}$$

$$\le B_m' \left(\frac{\mu_{g_0}(\Omega)}{\mu_g(\Omega)}\right)^{\frac{2}{m}}$$

With $B_m' = \frac{24c_m^2}{r_0^2}$, which completes the proof of Case 2.

In both cases, $\lambda_k(\Omega, g)$ is bounded above by the sum of the right-hand sides of (8) and (11), which completes the proof.

Theorem .5 There exist, for each integer $m \ge 2$, two constants A_m and B_m such that, for every compact Riemannian manifold (M, g) of dimension m and every $k \in N^*$, we have.

$$\lambda_k(M, g)\mu_g(M)^{\frac{2}{m}} \le A_m \mathrm{Vol}([g])^{\frac{2}{m}} + B_m k^{\frac{2}{m}}. \qquad (14)$$

It is important to notice that the constant B_m in inequality (14) cannot be equal to the constant α_m in the Weyl law. Indeed, it follows from [6. Corollary.1] that such a B_m must satisfy: $B_m \ge m w_m^{\frac{2}{m}}$. On the other hand, inequality (14) also gives an upper bound on the conformal spectrum introduced by [6] and shows that its asymptotic behavior obeys a Weyl type law.

Proof. Consider the $m-m$ space (M, d_{g_0}, μ_g), whered_{g_0} isthe distance associated with the metric g_0 and μ_g is the measure associated with the metric g. We easily see that we can follow the same arguments asin the proof of Theorem.4, to derive the following inequality.

$$\lambda_k(M,g)\mu_g(M)^{\frac{2}{m}} \leq A_m \mu_{g_0}(M)^{\frac{2}{m}} + B_m k^{\frac{2}{m}}. \qquad (15)$$

The left hand side does not depend on g_0. Hence, we can take the infimum with respect to $g_0 \in [g]$ such that $Ricci_{g_0} \geq -(m-1)$, which leads to the desired conclusion.

Now we give an application of Theorem.2.1 of [1], to the Steklov eigenvalue problem.

Let Ω be a bounded subdomain of a complete m-dimensional Riemannian manifold (M,g) and assume that Ω has nonempty smooth boundary $\partial\Omega$. Given a function $u \in H^{\frac{1}{2}}(\partial\Omega)$, we denote by $\bar{u}$ theunique harmonic extension of u to Ω, that is.

$$\begin{cases} \Delta_g \bar{u} = 0 & in\ \Omega \\ \bar{u} = u & on\partial\Omega \end{cases}$$

Let v be the outward unit normal vector along $\partial\Omega$. The Steklov operator isthe map.

$$L: H^{\frac{1}{2}}(\partial\Omega) \longrightarrow H^{-\frac{1}{2}}(\partial\Omega)$$

$$u \longmapsto \frac{\partial \bar{u}}{\partial v}$$

The operator L is an elliptic pseudo differential operator (see [7. Pages, 37-38]) which admits a discrete spectrum tending to infinity denoted by.

$$0 = \sigma_1 \leq \sigma_2 \leq \sigma_3 \ldots \nearrow \infty$$

The eigenvalue σ_k of L can be characterized variationally as follows (see[8]):

$$\sigma_k(\Omega) = \inf_{V_k} \sup \left\{ \frac{\int_\Omega |\nabla_g \bar{u}|^2 d\bar{\mu}_g}{\int_{\partial\Omega} |\bar{u}|^2 d\bar{\mu}_g} : 0 \neq \bar{u} \in V_k \right\}, \qquad (16)$$

where V_k is a k-dimensional linear subspace of $H^1(\Omega)$ and $\bar{\mu}_g$ is the Riemannian measure associated to g on the boundary.

The relationships between the geometry of the domain and the spectrum of the corresponding Steklov operator have been investigated by (see for example [8],[9] and [10]). Recently, [9] proved the following inequality for the Steklov eigenvalues of a compact Riemannian surface (Σ_γ, g) of genus γ and k boundary components:

$$\sigma_2(\Sigma_\gamma)\ell_g(\partial\Sigma_\gamma) \leq 2(\gamma + k)\pi,$$

Where $\ell_g(\partial\Sigma)$ is the length of the boundary. This result was generalized to higher eigenvalues by [8].Indeed, he the following inequality for every $k \in \mathbb{N}^*$.

$$\sigma_k(\Sigma_\gamma)\ell_g(\partial\Sigma_\gamma) \leq C(\gamma + 1)k, \qquad (17)$$

Where C is a universal constant.

For a domain in a higher dimensional manifold, [8] also obtained an upper bound for σ_k depending on the isoperimetric ratio of the domain. More precisely, if (M, g) is conformally equivalent to a complete manifold with non-negative Ricci curvature, then for every bounded domain Ω of $M$and every $k \in \mathbb{N}^*$,

$$\sigma_k(\Omega)\bar{\mu}_g(\partial\Omega)^{\frac{1}{m-1}} \leq C_m \frac{k^{\frac{2}{m}}}{I_g(\Omega)^{1-\frac{1}{m-1}}}, \qquad (18)$$

Where $I_g(\Omega)$ is the isoperimetric ratio $\left(I_g(\Omega) = \frac{\bar{\mu}_g(\partial\Omega)}{\mu_g(\Omega)^{\frac{m-1}{m}}}\right)$ and C_m is constant depending only on m.

The theorem below is motivated by the work of [8], and we obtain an improvement of inequalities (17) and (18) using Proposition. 3,

Corollary .6let $(\sum_\gamma, g)$ be a compact oriented Riemannian surface with genus γ, and Ω be a subdomain ofΣ_γ. Then

$$\sigma_k(\Omega)\ell_g(\partial\Omega) \leq A\gamma + Bk, \qquad (19)$$

Where $AandB$ are constants.

Theorem .7let(M, g_0) be a complete Riemannian manifold of dimension $m \geq 2$ with$Ricci_{g_0}(M) \geq -(m-1)$. Let $\Omega \subset M$ be a relatively compact domain with C^1 boundary and g be any metric conformal to g_0. Then wehave.

$$\sigma_k(\Omega)\bar{\mu}_g(\partial\Omega)^{\frac{1}{m-1}} \leq \frac{A_m \mu_{g_0}(\Omega)^{\frac{2}{m}} + B_m k^{\frac{2}{m}}}{I_g(\Omega)^{1-\frac{1}{m-1}}} \qquad (20)$$

Where A_m and B_m are constants depending only on m

Proof. We consider the $m-m$space $(\Omega, d_{g_0}, \bar{\mu})$, where $\bar{\mu}(A) := \bar{\mu}_g(A \cap \partial\Omega)$. We apply again Proposition.3, Therefore, there exist family of $3k$ capacitors$\{(F_i, G_i)\}$ satisfying properties (i), (ii) and either (iii)(a),or (iii)(b) of Proposition.3. We proceed analogously to the proof of Theorem.4. Using the variational characterization of σ_k, we construct a familyof test functions as in Case 1 and Case 2 of the proof of Theorem .4. In both cases, we have.

$$\sigma_k(\Omega) \leq \frac{\int_\Omega |\nabla_g f_i|^2 d\mu_g}{\int_{\partial\Omega} |f_i|^2 d\bar{\mu}_g} \leq \frac{\left(\int_\Omega |\nabla_{g_0} f|^m du_{g_0}\right)^{\frac{2}{m}} \mu_g(G_i)^{1-\frac{2}{m}}}{\bar{\mu}(F_i)}$$

If the family $\{(F_i, G_i)\}$ satisfies the properties (i) and (iii) (a) of Proposition.3, then.

$$\sigma_k(\Omega) \leq A_m \frac{\left(\frac{\mu_g(\Omega)}{k}\right)^{1-\frac{2}{m}}}{\frac{\bar{\mu}_g(\partial\Omega)}{k}} \leq A_m \frac{k^{\frac{2}{m}}}{\bar{\mu}_g(\partial\Omega)^{\frac{1}{m-1}} I_g(\Omega)^{1-\frac{1}{m-1}}} \qquad (21)$$

If on the other hand, the family $\{(F_i, G_i)\}$ satisfies the properties (i), (ii) and (iii) (b) of Proposition.3, then.

$$\sigma_k(\Omega) \leq B_m \frac{\left(\frac{\mu_{g_0}(\Omega)}{k}\right)^{\frac{2}{m}} \left(\frac{\mu_g(\Omega)}{k}\right)^{1-\frac{2}{m}}}{\frac{\bar{\mu}_g(\partial\Omega)}{k}} \leq B_m \frac{\mu_{g_0}(\Omega)^{\frac{2}{m}}}{\bar{\mu}_g(\partial\Omega)^{\frac{1}{m-1}} I_g(\Omega)^{1-\frac{1}{m-1}}} \qquad (22)$$

where the constant coefficients A_m and B_m are the same as A_m' andB_m' in Theorem.4.

The proof of inequalities (21) and (22) are along the same lines as Theorem.4. In both cases,$\sigma_k(\Omega)$ is bounded above by the sum on the right-hand sides of (21) and (22), and it completes the proof.

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