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Study of fundamental group and homotopy theory

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Abstract

This paper a short introduction to homotopy theory and its relation to algebraic topology. Algebraic topology tries to connect topological spaces with algebraical objects in such a way that topological problems can be translated into algebraical problems which can possibly be easier to solve. This paper is an introduction into the theory of homotopy and the basic concepts that concern it.

Keywords: Algebraic Topology, Topological problems, Theory of Homotopy.

1 Homotopic mappings:-The most important instance of a parametrized family of mappings (continuous understood) is obtained by taking the parameter space to be the closed unit interval I^1. plane E^2.Then any mapping $h:S^1\times I^1\rightarrow E^2$ is such a family. Each member $h|S^1\times t$, $0 \leq t \leq 1$, may be considered as a mapping of S^1 into E^2 and, in particular, the two members $h|S^1\times 0$ and $h|S^1\times 1$ may be viewed as continuous deformations of each other.

The mapping h is called a homotopy between f and g and the product space $X\times I^1$ is the homotopy cylinder.

The question of the existence of a homotopy between two mappings f, g:X→Y can be very difficult. The answer depends upon f and g, certainly, and also upon the structure of the spaces X and Y. It is evident that this question is one of extending a given mapping.

Two mappings of X into Y, then we have a mapping *h* on the closed subset $(X\times 0)\cup(X\times 1)$ of $X\times I^1$ given by $h'(x,0)=f(x)$ and $h'(x,1)=g(x)$. Then *f* and *g* are homotopic if and only if h can be extended to a mapping *h* of the entire product space $X\times I^1$ into *Y*. Thus it would seem that theorems about homotopy are but special cases of more general theorems on the extension of mappings. Indeed such is the case, but the general extension problem is far from being soled, and also the special case of homotopy plays an important role in the more general problem.

THEOREM 1.1 The homotopy classes of Y^X are precisely the arcwise connected components of γ^x.

Proof. This is merely a matter of checking definitions. For if $f\simeq g$, then the homotopy $h(x,t)$ between f and g defines a mapping $F:I^1\to Y^X$ given by

$$F(t)=f_t(x)=h(x,t).$$

Then $F(I^1)$ is a Peano continuum in Y^X, and as such contains an arc between *f* and *g*. Conversely, an "arc" of mapping between *f* and *g* provides a homotopy between the two.

THEOREM 1.2 Let *A* be a closed subset of a separable metric *M*, and let $f^`$ and $g^`$ be homotopic mappings of *A* into the *n*-sphere S^n. It there exists an extension *f* of $f^`$ to all of *M*, then there also exists an extension *g* of *g'* to all of *M*, and the extensions *f* and *g* may be chosen to be homotopic also.

Proof (we follow Dowker [74]): Let $h:A\times I^1\to S^n$ be the assumed homotopy between *f'* and *g'*, and let *f* be the given extension of $f^`$ to all of *M*. Let *D* be the set in $M\times I^1$ given by

$$D = (A \times I^1) \bigcup (M \times 0).$$

Clearly D is a closed subset of $M \times I^1$, and on D we may define the mapping $F : D \to S^n$ given by

$$F(x,0) = f(x) \quad \text{for all } x \text{ in } M,$$

and

$$F(x,t) = h\ (x,t) \quad \text{for all } x \text{ in } A \text{ and } 0 \le t \le 1.$$

Since $h\ (x,0) = f(x) = f(x)$ for all points x in A, this mapping F is well-defined and continuous.

2 **Homotopically equivalent spaces**. Two spaces X and Y are of the same homotpy type (are homotopically equivalent) if there exist mappings $f : X \to Y$ and $g : Y \to X$ such that the composite mappings $fg : Y \to Y$ and $gf : X \to X$ are homotopic, respectively, to the identity mappings $i : Y \to Y$ and $i : X \to X$. All the forthcoming algebraic groups to be associated with a space fail to distinguish between two homotopically equivalent spaces. It is obvious that homeomorphic spaces are of the same homotopy type, but the converse is not necessarily true. To give an example of general procedure for obtaining two homotopically equivalent spaces that are not homeomorphic, we prove a theorem.

Let $f : X \to Y$ be continuous. In the (disjoint) union $(X \times I^1) \cup Y$, identify each point $(x,1)$ with the point $f(x)$ in Y. Using the identification topology, the resulting space $Y_{f(X)}$ is called the mapping cylinder of f. As a special case, if $c : X \to p$ is a constant mapping of X onto a space with only one point p, then the mapping cylinder of c is homeomorphic to the join pX.

THEOREM2.1 Let $f : X \to Y$ be any continuous mapping of a space X into a space Y. Then the mapping cylinder $Y_{f(X)}$ is homotopically equivalent to Y.

Proof: Define a mapping $g : Y_{f(X)} \to Y$ by setting

$f(x,t) = f(x)$ for (x,t) in $X \times I^1$

And

$g(y) = y$ for y in Y.

This mapping is well-defined and continuous on $Y_{f(X)}$ because it is continuous on each of two closed subsets of $Y_{f(X)}$ and agrees on the inter section of these subsets. Next let $h: Y \to Y_{f(X)}$ be the identity injection $h(y) = y$. Clearly we have

$$gh(y) = g(y) = y,$$

so the composite mapping $gh: Y \to Y$ is the identity mapping. Considering the composite mapping hg of $Y_{f(X)}$ into itself, we have

$hg(y) = h(y) = y$ for all points y in Y

And

$hg(x,t) = h(f(x)) = f(x)$ for all points (x,t) in $X \times I^1$

We define a mapping $H: Y_{f(X)} \times I^1 \to Y_{f(X)}$ by setting

$H(y,s) = y$ for all y in Y and $0 \leq s \leq 1$

$H((xt),s) = (x,(1-s)t + s)$ for (x,t) in $X \times I^1$ and $0 \leq s \leq 1$.

Went t = 1, we have

$$H((x,1),s) = (x,1) = f(x) = H(f(x),s) \quad (0 \leq s \leq 1),$$

So the two definitions agree on those points identified in $Y_{f(x).}$ Hence H is well-defined and continuous. But now

$$H(y,0) = y,$$

$$H((x,t),0) = (x,t),$$

or $H(z,0)$ is the identity mapping on $Y_{f(X),}$ while

$$H(y,1) = y$$

And

$$H((x,t),1) = (x,1) = f(x),$$

so $H(z,1) = hg(z)$ for all points z in $Y_{f(X)}$. Therefore H is a homotopy between the identity mapping on $Y_{f(X)}$ and the composite mapping hg.

We can state a corollary to Theorem 3 by giving another definition. A subset D of a space X is a deformation retract of X if there is a retraction r of X onto D which is homotopic to the identity mapping of X onto itself under a homotpy that leaves D fixed. That is, there is a homotopy $h: X \times I^1 \to X$ such that

$$h(x,0) = x \qquad \text{for all } x \text{ in } X,$$

$$h(x,1) = r(x) \qquad \text{for all } x \text{ in } X,$$

and

$$h(x,t) = x \qquad \text{for all } x \text{ in } D \text{ and } 0 \le t \le 1.$$

COROLARY The space Y is a deformation retract of the mapping cylinder $Y_{f(X)}$.

***Proof*:** Consider the mapping $g: Y_{f(X)} \to Y$ given in the proof of Theorem 2.1 Clearly $g(y) = y$ for each point y in Y, so g is a retraction of $Y_{f(X)}$ onto Y. The homotopy $H(z,s)$ given in Theorem 2.9 has the property that

$$H(z,0) = z$$

and

$$H(z,1) = g(z).$$

Thus$H$ is a homotopy between the identity mapping on $Y_{f(X)}$ and the mapping g. Finally, for any point y in Y, we have

$$H(y,s) = y,$$

3 The Fundamental Group

The set of path-homotopy classes of paths in a space X does not form a group under the operation because the product of two path-homotopy classes is not always defined. But suppose we pick out a point x_0 of X to serve a “base point” and restrict ourselves to those paths that begins and end at x_0. The set of these path-homotopy classes does form a group under. It will be called the fundamental group of X.

Definition: Let X be a space; let x_0 be a point of X. A path in X that begins and ends at x_0 is called a loop based at x_0. The set of pathhomotopy classes of loops based at x_0, with the operation, is called the *fundamental group* of X relative to the base point x_0. It is denoted $\pi_1(X,x_0)$

Given two loops f and g based at x_0, the product $f*g$ is always defined and is a loop based at x_0. Associatively, the existence of an identity element $[e_{x_0}]$, and the existence of an inverse $[\overline{f}]$ for $[f]$ are immediate.

Sometimes this group is called the first homotopy group of X, which term implies that there is a second homotopy group. There are indeed groups $\pi_n(X,x_0)$ for all $n \in \square_{+} i$

Definition: Let α be a path in X from x_0 to x_1.we define a map

$$\hat{\alpha}:\pi_1(X,x_0)\rightarrow\pi_1(X,x_1)$$

by the equation

$$\hat{\alpha}:([f])=[\hat{\alpha}]*[f]*[\alpha]$$

The map $\hat{\alpha}$ which we call "$\alpha-$hat,"is well-defined because the operation * is well-defined. If f is a loop based at x_0, then $\hat{\alpha}*(f*\alpha)$ is a loop based at x_1. Hence $\hat{\alpha}$ maps $\pi_1(X,x_0)$, into $\pi_1(X,x_1)$, as desired; note that it depends only on the path-homotopy class of α.It is pictured in Figure 3.1.

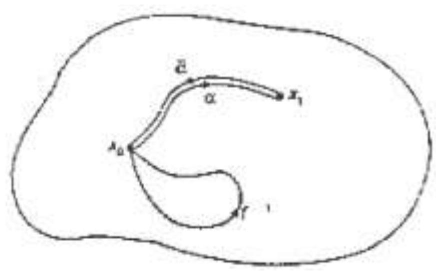


Figure 3.1

Theorem 3.1 The map $\hat{\alpha}$ is a group isomorphism.

Proof. To show that $\hat{\alpha}$ is a homomorphism, we compute

$$\hat{\alpha}[f]*\hat{\alpha}([g]) = ([\hat{\alpha}]*[f]*[\alpha])*[(\hat{\alpha}]*[g]*[\alpha])$$

$$= [\hat{\alpha}]*[f]*[g]*[\alpha]$$

$$= \hat{\alpha}([f]*[g]).$$

To show that $\hat{\alpha}$ is an isomorphism, we show that if β denotes that path $\hat{\alpha}$, which is the reverse of α, then $\hat{\beta}$ is an inverse for $\hat{\alpha}$. We compute, for each element [h] of $\pi_1(X,x_1)$.

$$\hat{\beta}([h]) - [\bar{\beta}]*[h]*[\beta] = [\alpha]*[h]*[\hat{\alpha}],$$

$$\hat{\alpha}\,\hat{\beta}([h]) = [\hat{\alpha}]*[\alpha]*[h] = [\hat{\alpha}])*[\alpha] = [h],$$

A similar computation shows that $\hat{\beta}(\hat{\alpha}([f]) = [f]$ for each $[f] \in \pi_1(X,x_0)$

Corollary If X is path connected and x_0 and x_1 are two points of X, then $\pi_1(X,x_0)$ is isomorphic to $\pi_1(X,x_1)$

Suppose that X is a topological space. Let C be the path component of X containing x_0. It is easy to see that $\pi_1(C,x_0) = \pi_1(X,x_0)$, since all loops and homotopies in X that are based at x_0 must lie in the subspace C. Thus $\pi_1(X,x_0)$ depends on only the path component of X containing x_0; it gives us no information whatever about the rest of X. For this reason, it is usual to deal with only path-connected spaces when studying the fundamental group.

It X is path connected, all the groups $\pi_1(X,x)$ are isomorphic, so it is temping to try to "identify" all those groups with one another and to speak simply of the fundamental group of the space X, without reference to base point. The difficulty with this approach is that there is no natural way of identifying $\pi_1(X,x_0)$ with $\pi_1(X,x_1)$; different paths α and β from x_0 to x_i may give rise to different isomorphism between these groups. For this reason, omotting the base point can lead to error.

It turns out that the isomorphism of $\pi_1(X,x_0)$ with $\pi_1(X,x_1)$ is idependent of path if and only if the fundamental group is abelian. This is a stringent requirement on the space X.

Definition Let $p: E \to B$ be a covering map; let $b_0 \in$ B. Choose e_0 so that $p(e_0) = b_0$. Given an element $[f]$ of π_1 (B, b_0), let $\tilde{f}$ be the lifting of f to a path in E that begins at e_0: Let $\phi([f])$ denote the end point $\tilde{f}(1)$ of $\tilde{f}$. Then ϕ is a well-defined set map.

$$\phi: \pi_1(B,b_0) \to p^{-1}(b_0)$$

We call ϕ the *lifting correspondence* derived from the covering map p. It depends of course on the choice of the point e_0.

Theorem 3.2 Let $p: E \to B$ be a covering map; let $p(e_0) = b_0$. If E is path connected, then the lifting correspondence.

$$\phi: \pi_1(B,b_0) \to p^{-1}(b_0)$$

is surjective. It E is simply connected, it is bijective.

Proof. If E is path connected, then, given $e_1 \in p^{-1}(b_0)$, there is a path $\tilde{f}$ in E from e_0 to e_1. Then $f = p \circ \tilde{f}$ is a loop in B at b_0, and $\phi([f]) = e_1$ by definition.

Supoose E is simply connected. Let [*f*] and [*g*] be two elements of $\pi_1(B,b_0)$ such that $\phi([f]) = \phi([g])$. Let $\tilde{f}$ and $\tilde{g}$ be the lifting of f and g, respectively, to paths in E that begin at e_0; then $\tilde{f}(1) = \tilde{g}(1)$. Since E is simply connected, there is a path homotopy $\tilde{F}$ in E between $\tilde{f}$ and $\tilde{g}$. Then $p \circ \tilde{F}$ is a path homotopy in B between f and g.

Theorem 3.3 Let $p: E \to B$ and $p': E' \to B$ be covering maps; let $p(e_0) = p'(e'_0) = b_0$. There is an equivalence $h: E \to E'$ such that $h(e_0) = e'_0$ if and only if the groups

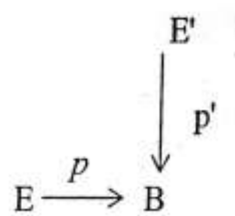
$$H_0 = p_*(\pi_1(E,e_0)) \text{ and } H'_0 = p'_*(\pi_1(E',e'_0))$$

are equal. If *h* exists it is unique.

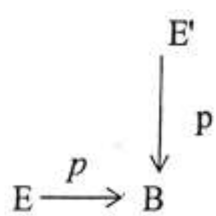
Proof. We prove the “only if” part of theorem. Given h, the fact that h is a homemorphism implies that $h_*(\prod_1(E,e_0))=\prod_1\ (\mathrm{E}^1,e_0^1)$

Since $p'\ \mathrm{o}\ h=p,$ we have $\mathrm{H}_0=\mathrm{H'}_0$

Now we prove that "If "part of the theorem: we assume that $\mathrm{H}_0=\mathrm{H'}_0$ and show that h exists. We shall apply the preceding lemma. Consider the maps



Because p' is a covering map and E is path connected and locally path connected there exists a map $h:\mathrm{E}\rightarrow\mathrm{E'}$ with $h(e_0)=e'_0$ that is a lifting of p (that is, such that $p'\ \mathrm{oh}=p$). Reversing the roles of E and E' in this argument, we see there is a map k : $E'\rightarrow E$ with $k(e'_0)=e_0$ such that $p\ \mathrm{o}\ k=p'$.Now consider the maps



The map $k\ \mathrm{o}\ \mathrm{h}:\mathrm{E}\rightarrow\mathrm{E}$ is a lifting of P (since $p\ \mathrm{o}\ k\ \mathrm{o}\ h=p'\mathrm{o}\ h=p),$)with $\mathrm{p}(\mathrm{e}_0)=\mathrm{e}_0$. The identity map i_E of E is another such lifting. The uniquencess part of the preceding lemma implies that $k\ o\ h=\mathrm{i}_\mathrm{E}$. A similar argument shows that $h\ o\ k$ equals the identity map of E'.

We seem to have solved our equivalence problem. But there is a somewhat suitable point we have overlooked. We have obtained a necessary and sufficient condition for there to exist an equivalence $h:\mathrm{E}\rightarrow\mathrm{E'}$ that carries the point e_0 to the point $\mathrm{e'}_0$. But we have not yet determined under what conditions there exists an equivalence in general. It is possible that there may be no equivalence carrying e_0 to e'_0 but that there is an equivalence carrying e_0 to some other point e'_1 of $(p')^{-1}(b_0)$. Can we determine whether this is the case merely by examining the subgroups H_0 and H'_0? We consider this problem now.

It H_1 and H_2 are subgroups of a group G, you may recall from algebra that they are said to be conjugate subgroups if $H_2 = \alpha . H_1 \alpha^{-1}$ for some element α of G. Said differently, they are conjugate if the isomorphism of G with itself that maps x to $\alpha . x . \alpha^{-1}$ carries the groupo H_1 onto the group H_2. It is easy to check that conjugacy is an equivalence relation on the collection of subgroups of G. The equivalence class of the subgroup H is called the *conjugacy class* of H.

BIBLIOGRAPHY

1. Dr. B.K. Singh & G. Kumar: Some contribution to fundamental group &Homotopy theory ARJPS, Vol.17, P.13-19 (2014)

2. Cohn, P.M. : Free Rings and their Relations, Academic Press, New York, 1971.

3. Anderson, F. & Fuller, K.: Rings and Categories of Module. Springer-verlag, Berlin, 1974.

4. North Cott, D.G.: An Introduction to Homological Algebra, Cambridge Univ. Press, London & New York, 1960.

5. Mac Lane S. Homology New York, Berlin, Gottingen and Heideberg: Springer, 1963.

6. Cartan, H. and S. Eilenberg, Homological Algebra, Princeton; Princeton University Press, 1956.

7. Auslander, M., and D.A. Buchsbaum, "Homological dimension in Noetherian Rings II". Trans. Amer, Math. Soc. Vol. 88. (1958) pp. 194-206.

8. Bass, H. "Finitistic dimension and a homological generalization of semi-primary rings". Trans. Amer. Math. Soc, Vol. 95 (1960), pp. 446-488.

9. D.W. Hall and G.L. Spencer. Elementary Topology, John Wiley & Sons. Inc. New York 1955.

10. J.L. Kelley, General Topology Springer-Verlag, New York 1961.

11. D. Montgomery and L. Zippin, Topological Transformation Groups. Inter Science Publishers. Inc. New York, 1955.

12. James R. Munkres. Topology Low Price Edition.

13. G.F. Simmons. Introduction to Topology and Modern Analysis. Tata Mc.Grow Hill Edition.

14. Dr. K.K. Jha, Advanced General Topology. Nav Bharat Publication Delhi-6.

15. Alexandroff, P., Combinatorial Topology, Vol I. Rochester N.Y. Graylock, 1956.

16. Hilton, P.J., An Introduction to Homotopy theory Cambridge. Cambridge University Press, 1953.

17. Hurewiez, S.T. Homotopy Theory, New York: Academic Press, 1959.

18. Kelly J.L. General Topology, New York; Van Nostrand, 1955.

19. Eilenberg, S. "Singular homology theory" Ann. Math 45, 407-447 (1944).

20. Taylor, A.E.: "Introduction to Functional Analysis" Wiley, New York, 1958.

21. Riesz, F., and B. S. Nagy "Functional Analysis" Frederick Ungar, New York, 1995.

22. Stone M.N. : On the Compactification of Topological Spaces, Ann. Soc. Pol. Math: 21 (1948), pp. 153-160.

23. John G. Hocking: "Topology" Addition-Wesley Publishing Company, Inc. London, U.S.A. England.

24. Samuel I. Goldberg, Curvature and Homology. Academic Press New York and London 1962.

25. Joseph J. Rotman : An Introduction to homological Algebra. Academic Press New York, 1979.

26. Holt, Rineharta Winston; "Rings and Homology" New York, Chicago, San Francisco, 1964.