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EXACT COUPLES AND HOMOLOGY OF FILTRATIONS

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Abstract

In this paper we study the relationship between a very classical algebraic object associated to a filtration of topological spaces, namely a spectral sequence, and a more recently invented object that has found many applications – namely, its persistent homology groups. We show the existence of a long exact sequence of groups linking these two objects and using it derive formulas expressing the dimensions of each individual groups of one object in terms of the dimensions of the groups in the other object. The main tool used to mediate between these objects is the notion of exact couples first introduced by Massey in 1952.

Key words: Homology, Cohomology, topological spaces, spectral sequence

Introduction

A very classical technique in algebraic topology for computing topological invariants of a space X is to consider a filtration F of X where the successive spaces FsX capture progressively more and more of the topology of X. For example, in case X is a CW-complex one can take for FpX the p-th skeleton skp(X) consisting of all cells of dimension at most p. More generally, given a cellular map $f : X \to Y$, one can take for FpX the inverse image under f of skp(Y). One then associates to this sequence a sequence of algebraic objects which in nice situations is expected to "converge" (in an appropriate sense) to the topological invariant (such as the homology or cohomology groups) associated to X itself, directly computing which is often an intractable problem. This sequence of algebraic approximations is called a spectral sequence associated to the filtration F, and was first introduced by Leray [12] in 1946. Spectral sequences are now ubiquitous in mathematics. A typical application which is common in discrete geometry, as well as in quantitative real algebraic geometry, is to use the initial terms of a certain spectral sequence to give upper bounds on the topological complexity (for example, the sum of Betti numbers) of the object of interest X.

EXACT COUPLES AND FIVE-TERM SEQUENCES

Definition 1: A graded module is a sequence of modules M = $\{M_p : p \in Z\}$.If $M = \{M_p\}$ and $N = \{N_p\}$ are graded modules and a is a fixed integer, then a sequence of homomorphisms $f = \{f_p : M_p \to N_{p+a}\}$ is a map of degree a. One writes $f : M \to N$.

A complex $C = \ldots \to C_p \xrightarrow{d_p} C_{p-1} \to \ldots$ determines a graded module $C = \{C_p : p \in Z\}$ if one ignores the differentiation $d = \{d_p : p \in Z\}$.The map $d : C \to C$ has degree-1. If (C', d') is another complex, a chain map $f : C \to C'$ gives a map of degree 0 (with *fd=df'*), while a homotopy is a certain type of map of degree +1. A second example of a graded module is the homology of a complex $C : H_*(C) = \{H_p(C) : p \in Z\}$. One may reverse this procedure: given a graded module $\{A_p : p \in Z\}$, define a complex

$$A = \ldots \to A_p \xrightarrow{d_p} A_{p-1} \to \ldots$$

in which each $d_p = 0$; one says A is a complex with zero differentiation.

Exercises: Degrees add under composition: if $f: M \to N$ has degree a and $g: N \to K$ has degree b, then $gf: M \to K$ is a map of degree $a+b$.

All graded modules (over a fixed ring) and all maps having a degree comprise a category. Note that

$$\text{Hom}(M,N) = \bigcup_{a \in Z} \left(\prod_p \text{Hom}(M_p, N_{p-a}) \right)$$

Recall a mnemonic introduced when we first saw long exact sequences: exact triangles. If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence of complexes, then the long exact sequence may be written

$$\begin{tikzcd}
H_*(A) \arrow[rr, "f_*"] & & H_*(B) \arrow[dl, "g_*"] \\
& H_*(C) \arrow[ul, "\partial"] &
\end{tikzcd}$$

Regarding the vertices as graded modules, the maps f_* and g_* have degree 0 and ∂ has degree-1. Conversely, given any exact triangle, one may write down a long exact sequence of he knows the degrees of the maps.

Definition 2: A **bigraded module** is a doubly indexed family of modules $M = \{M_{p,q} : (p,q) \in Z \times Z\}$.If $M = \{M_{p.q}\}$ and $N = \{N_{p.q}\}$ are bigraded modules and if (a,b) is a fixed ordered pair of integers, then a family of homomorphism $f = \{f'_{p.q} : M_{p.q} \to N_{p+a,q+b}\}$ is a map of bidegre (a,b). One writes $f: M \to N$.

Exercises: Bidegrees add under composition: if $f : M \to N$ has bidegree (a,b) and $g : N \to K$ has bidegree $(a^{'},b^{'})$, then $gf : M \to K$ has bidegree $(a+a^{'},b+b^{'})$.

All bigraded modules (over a fixed ring) and all maps having a bidegree comprise a category.

In the category of bigraded modules, there are subobjects and quotient objects. If $M_{p.q} \subset N_{p.q}$ for all $p,q,$ then $M = \{M_{p.q}\}$ is a (bigraded) submodule of $N = \{N_{p.q}\}$; visibly, the inclusion map $M \to N$ has bidegree (0,0). Define the (bigraded) quotient module *N/M* as $\{N_{p.q}/M_{p.q}\}$; the natural map $N \to N/M$ also has bidegree (0,0).

There is a consequence of this elementary definition. Given $f : M \to N$ with bidegree (a,b), im*f* should be a (bigraded) submodule of *N*; what is $(\mathrm{im}f)_{p.q}$? Since $(imf)_{p.g} \subset N_{p.q}$, we are forced to define

$$(imf)_{p.q} = f_{p-a,q-b}(M_{p-a,q-b}) = im\ (f_{p-a,q-b}) \subset N_{p.q.}$$

(Thus, $(\mathrm{im}f)_{p.q}$ is not int $f_{p.q}$), which lies in $N_{p+a.q+b).}$ On the other hand, there is no problem with indices of $\ker f$:if one defines

$$(\ker f)_{p.q} = \ker(f_{p,q}),$$

then $\ker f$ is a (bigraded) submodule of *M*. It is now clear how to define exactness of a sequence of bigraded modules.

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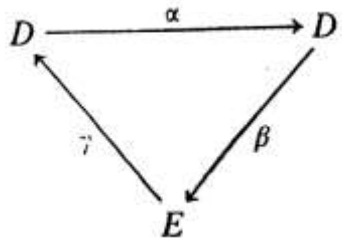
We defer giving examples of bigraded modules; later, we shall define a "bicomplex" which will determine bigraded modules in much the same way that a comlex determines graded modules (ignore differentiation; homology).

Definition 3: An exact couple is a pair of bigraded modules D and E, and maps x, β, γ (each of some bidegree) such that there is exactness at each vertex of the triangle



Obviously, an exact couple generalizes the notion of exact triangle (since one does not generalize merely for the sake of generalization, there are not three distinct bigraded modules at thevertices; in practice, an exact couple is what one encounters). Given an exact couple $D = \{D_{p,q}\}, E = \{E_{p,q}\}$, and maps x, β, γ of bidegress $(a, a^{'}), (b, b^{'}), (c, c^{'})$, respectively, one may write down a long exact sequence for each fixed q:

$$\ldots \xrightarrow{\beta} E_{p-c,q-c} \xrightarrow{i} D_{p.q.} \xrightarrow{x} D_{p-a,q+a} \xrightarrow{\beta} E_{p+a+b,q+a+b} \xrightarrow{\beta} \ldots$$

Conversely, given infinitely many long exact sequences as above, they may be assembled into one exact couple.

Let us now use this barrage of notation (actually, a clever organization of a maze of data, due to Massey) to obtain concrete results. The next theorem is due to Kreimer, while the notion of acyclicity when dealing with compositefunctors was first formulated by Grothendieck.

We maintain our (too restrictive) hypothesis that all functors are additive functors between categories of modules (one may replace module categories by certain more general categories).

Definition 4: Let $B :\to R$ be a functor of either variance. A module B in $\Re$ is right F-acyclic if $(R^p F)B = 0$ for all $p \geq 1$, where $R^p F$ is the pth right derived functor of F: a module B in $\Re$ is left F-acyclic if $(L_p F(B = 0$ for all $p \geq 1$. where $L_p F$ is the pth left derived functor of F.

If F is covariant, recall that $(R^P F)B = H^p(FE_B)$, where E_B is a deleted injective resolution of B. It follows from that every injective module is right F-acyclic.

Exercises: Every projective module is left F-acyclic for any co-variant functor$F$, and is right F-acyclic for any contravariant functor$F$.

If $F = A \otimes_R$, then every flat module ${}_R B$ is left F-acyclic.

A composite of functors may give an exact couple. Observe first that if $G:\mathfrak{R}\to\aleph$, then $(R^nG)A\in\mathfrak{R}$ for every module A in u(if $\ldots\to X_n \xrightarrow{Gd_n} GX_{n-1}\to\ldots$ is a complex in N, whence $\ker Gd_n/\operatorname{im} Gd_{n+1}\in\mathfrak{R}$).

Theorem 5: *Let $E:R\to B$ and $F:B\to C$ be functors such that F is left exact and whenever E is injective in R, then GE is right F-acyclic. For each module A in R, choose an injective resolution $0\to A\to E^0\to E^1\to\ldots$ and define*

$$Z^q=\ker(GE^q\to GE^{q+1}).$$

Then there exists an exact couple with

$$E_{p.q}=\begin{cases}(R^PF)(R^qG(A)) & \text{if } p\geq 0, q\geq 0.\\ 0 & otherwise\end{cases}$$

$$D_{p,q}=\begin{cases}(R^pF)Z^{q-1} & \text{if} \quad p\geq 0, q\geq 1,\\ R^{p+q}(FG)A & \text{if} \quad p=1, q\geq 1,\\ 0 & otherwise,\end{cases}$$

and maps $\alpha:D\to D$ of bidegree $(-1,1)$, $\beta:D\to E$ of bidegree (1,-1), and $y:E\to D$ $E\to D$ of bidegree $(1,0)$.

Remarks: 1. Visualize a bigraded module as a family of modules, one sitting on each lattice point in the *p-q* plane. Thus, *E* lives in the first quadrant and *D* lives above the line *q*=1 and to the right of *p*=-1.

2. The basic idea is just to assemble the long exact sequences arising from the obvious short exact sequences [1] and [4] below.

***Proof*:** Abbreviate $R^pG, R^pF, R^p(FG)$ to $G^p, F^p, (FG)^p$, respectively. Our task is to exhibit, for each $q \geq 0$, an exact sequence

$$E_{-1.q-1} \to D_{0.q+1} \to D_{-1.q+2} \to E_{0.q-1} \to \ldots$$
$$\to D_{p.q+1} \to D_{p-1,q+2} \to E_{p.q+1} \to \ldots$$

By definition of E and D, we want exact sequences

$$0 \to F^1Z^q \to (FG)^{q+1}A \to F(G^{q+1}A) \to \ldots$$

$$\to F^{p+1}Z^q \to F^pZ^{q+1} \to F^p(G^{p+1}A) \to \ldots$$

Now G^qA is just the qth homology of the complex $\ldots \to GE^q \to GE^{q+1} \to \ldots$. We have already denoted the q-cycles of this complex by Z^q; denote the q-boundaries, $\operatorname{im}(GE^{q-1} \to GE^q)$, by B^q. There are short exact sequences

[1] $$0 \to Z^q \to GE^q \xrightarrow{\lambda} B^{q+1} \to 0$$

which gives rise to exact sequences (since F is left exact and GE^q is right F-acyclic)

[2] $$0 \to FZ^q \to FGE^q \to FB^{q+1} \to F^1Z^q \to 0, \qquad \text{all} \quad q \geq 0,$$

and isomorphisms

[3] $$F^pB^{q+1} \xrightarrow{\sim} F^{p+1}Z^q, \qquad \text{all } p \geq 1. \quad q \geq 0.$$

The definition of homology gives short exact sequences

[4] $$0 \to B^{q+1} \xrightarrow{\mu} Z^{q+1} \xrightarrow{\nu} G^{q+1}A \to 0.$$

which yields long exact sequences

$$0 \to FB^{q+1} \to FZ^{q+1} \to F(G^{q+1}A) \to F^1B^{q+1} \to F^1Z^{q+1} \to F^1(G^{q+1}A) \to \ldots$$

Using [3] to replace each term F^pB^{q+1} by its isomorphic copy $F^{p+1}Z^q$, for $p \geq 1$, almost leaves us with the desired exact sequences; only the first two terms are not correct.

Consider the diagram

$$\begin{tikzcd}
 & FGE^q \arrow[r, equal] \arrow[d, "F\lambda"] & FGE^q \arrow[d, "h"] & & & \\
0 \arrow[r] & FB^{q+1} \arrow[r, "F\mu"] \arrow[d] & FZ^{q+1} \arrow[r, "F\nu"] \arrow[d] & F(G^{q+1}A) \arrow[r] \arrow[d, equal] & F^2Z^q \arrow[r] & \cdots \\
0 \arrow[r] & F^1Z^q \arrow[r, dashed] \arrow[d] & W \arrow[r, dashed] \arrow[d] & F(G^{q+1}A) \arrow[r] & F^2Z^q \arrow[r] & \cdots \\
 & 0 & 0 & & &
\end{tikzcd}$$

where the first column is the end of the exact sequence [2], the map h is the composite $F\mu F\lambda = F(GE^q \to B^{q+1} \to Z^{q+1})$, and $W = co\ker h$.Commutativity of the top square and exactness of the columns provide a unique map $F^1Z^q \to W$ making the bottom square commute; that $F\nu \circ h = 0$ implies the existence of a map $W \to F(G^{q+1}A)$ which makes the remaining square commute. Exactness of the middle row implies, by diagramchasing exactness of the bottom row. It remains to identify W with $(FG)^{q+1}A$.

Now $Z^{q+1} = \ker(GE^{q+1} \to GE^{q+2})$.so that left exactness of F gives $FZ^{q+1} = \ker(FGE^{q+1} \to FGE^{q+2})$.Therefor

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$$W = \mathrm{coker}(FGE^q \to FZ^{q+1} = FZ^{q+1} / \mathrm{im}(FGE^q \to FZ^{q+1})$$

$$= \ker(FGE^{q+1} \to FGE^{q+2}) / \mathrm{im}(FGE^q \to FZ^{q+1}).$$

Using left exactness of F once again. $Z^{q+1} \to GE^{q+1}$ implies $FZ^{q+1} \to FGE^{q+1}$, and thus $\mathrm{im}(FGE^q \to FZ^{q+1}) = \mathrm{im}(FGE^q \to FZ^{q+1})$. Therefore,

$W = H^{q+1}(FGE_A) = (FG)^{q+1} A.$

Theorem6.(Cohomology Five-Term Sequence): *Let* $G : U \to B$ and $F : B \to C$ *be left exact functors such that E injective in U implies GE is right F-acyclic. Then there is an exact sequence*

$$0 \to (R^1F)(GA) \to R^1(FG)A \to F(R^1GA) \to (R^2F)(GA) \to R^2(FG)A$$

for every module A in U.

Proof: We return to the abbreviated notation above. Consider the exact sequences in for $q = 0$ and $q = 1$:

$$0 \to F^1Z^0 \to (FG)^1 A \to F(G^1A) \to F^2GA \to F^1Z^1 \to \ldots$$

and

$$0 \to F^1Z^1 \to (FG)^2 A \to \ldots$$

Just splice these two sequences together at F^1Z^1, and remember that $Z^0 - \ker(GE^0 \to GE^1) = GA$, for G is left exact.

Remark: If G is not left exact, one must replace GA by $(R^0G)A$ in the sequence.

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Theorem 7. *Let $G : U \to B$ and $F : B \to C$ be leftexact functors such that E injective in U implies GE is right F-acyclic. If A is a module in U with $(R^i G)A = 0$ for $1 \le i < q$, then there is an exact sequence*

$$0 \to (R^q F)(GA) \to R^q(FG)A \to F(R^q GA) \to (R^{q-1}F)(GA) \to R^{q+1}(FG)A.$$

Proof: Let us return once again to the abbreviated notation for derived factors. We shall only prove the special (though most important) case when $G^1 A = 0$ (the rest being an exercise for the reader). Consider the exact sequences in Theorem 11.1 for $q = 0,1,$ and 2:

$$0 \to F^1 Z^0 \to (FG)^1 A \to F(G^1 A) \to F^2 Z^0 \to F^1 Z^1 \to F^1(G^1 A) \to \ldots,$$

$$0 \to F^1 Z^1 \to (FG)^2 A \to F(G^2 A) \to F^2 Z^1 \to F^1 Z^2 \to \ldots,$$

and

$$0 \to F^1 Z^2 \to (FG)^3 A \to \ldots$$

Splice the last two sequences together at $F^1 Z^2$ to obtain exactness of

$$0 \to F^1 Z^1 \to (FG)^2 A \to F(G^2 A) \to F^2 Z^1 \to (FG)^3 A.$$

Since $G^1 A = 0,$ the first sequence $(q = 0)$ gives an isomorphism

$$F^2 Z^0 \cong F^1 Z^1.$$

Recalling that $Z^0 = GA,$ the sequence now begins with $F^2(GA).$ Let us deal with the fourth term $F^2 Z^1.$ Since $G^1 A = Z^1 / B^1,$ the hypothesis gives $Z^1 = B^1$ and hence $F^2 Z^1 = F^2 B^1.$ But isomorphism [3] gives $F^2 B^1 \cong F^3 Z^0 = F^3 GA.$

An important use of this last result occurs in the cohomology of algebras, involving Hilbert's "Theorem 90" and the Brauer group.

It should be clear that the proofs just given dualize: we merely state the results for right exact covariant functors.

Theorem 8. (**Homology Five-Term Sequence**): *Let* $G: U \to B$ *and* $F: B \to C$ *be right exact functors such that P projective in U implies GP is left F-acyclic. Then there is an exact sequence*

$$L_2(FG)A \to (L_2F)(GA) \to F(L_1GA) \to (FG)A \to (L_1G)(GA) \to 0$$

for every module A in U. Moreover, if $(L_iG)A = 0$ for $1 \le i < q$*.then there is an exact sequence*

$$L_{q-1}(FG)A \to (L_{q+1}F)(GA) \to F(L_qGA) \to L_q(FG)A \to (L_qF)(G.A) \to 0.$$

Let us illustrate these general results. Assume $\prod$ is a group with normal subgroup *N*. Clearly, every $\prod$-module *A* may be regarded as an *N*-module, so that Hom_N (Z,*A*) is defined. Recall

$$\mathrm{Hom}_N(Z, A) = A^N = \{a \in A : n\ a = a \text{ for all } n \in N\}.$$

For a $\prod$-module *A*, the module A^N is actually a $\prod / N-\text{module}: \text{if } x \in \prod$ has coset $\overline{X} \in \prod / N$, define

$$\overline{X}.a = x.a, \quad a \in A^N$$

(one checks this is well defined). Therefore Hom_N (Z,), and its derived functors: $\mathrm{Ext}^i_{Z_N}(Z,) = H^i(N,)$ are functors from $\prod$ -modules to $\prod$/*N*-modules.

What is the action of $\prod / N$ on $H^i(N, A)$ for a $\prod$-module A? Take a $\prod-$ projective resolution of Z

$$\mathrm{P} = \ldots \to P_1 \to P_0 \to Z \to 0$$

(which is automatically N-projective) and let us compute $H^i(N, A)$ by examining the complex $\mathrm{Hom}_N(P_Z, A)$. tells us a reasonable way to make each $\mathrm{Hom}_N(P_i, A)$ into a $\prod / N -$module.

$$(\overline{Xf})(b) = x.f(x^{-1}b), \quad \bar{x} \in \prod / N, \quad b \in P_i.$$

Hence, $H^i(N, A)$ becomes a $\prod / N -$ module by

$$\overline{X}(z_i + B_i) = \overline{X}.z_i + B_i,$$

Where $z_i \in \mathrm{Hom}_N(P_i, A)$ is a cycle and B_i is the submodule of boundaries. (Observe that the action of coincides with the action of $\prod / N$ on Hom_N (Z,A) given above.)

A similar discussion shows how $\prod / N$ acts on the homology groups $H_i(N, A)$ when A is $\prod-$module:

$$\overline{X}(b \otimes a) = bx^{-1} \otimes xa, \qquad \overline{X} \in \prod / N, \qquad b \in P_i.$$

Remark: One may prove [Gruenberg,1970,p.151] that if $N \subset Z(\prod)$ and A is $\prod-$trivial, then $H^p(N.A)$ and $H_p(N, A)$ are $\prod / N -$trivial for all p. The proof of this is not difficult, but involves a longish digression. This fact will be used several times in the sequel.

Theorem 9: *If N is normal subgroup of $\prod$ and A is a $\prod$-module, then there is an exact sequence*

$$0 \to H^1(\prod/N, A^N) \to H^1(\prod, A) \to H^1(N, A)^{\Pi/N} \to H^2(\prod/N, A^N) \to H^2(\prod, A).$$

Moreover, if $H^i(N, A) = 0$ for $1 \le i \le q$, *there is an exact sequence*

$$0 \to H^q(\prod/N, A^N) \to H^q(\prod.A) \to H^q(N, A)^{\Pi/N} \to H^{q+1}(\prod/N, A^N) \to H^{q+1}(\prod, A).$$

Proof: Let U be the category of $\prod$-modules and B the category of $\prod/N-$ modules: define $G: U \to B$ by $G = Hom_N(Z,)$ and $F: B \to \mathrm{Ab}$ by $F = \mathrm{Hom}_{\prod/N}(Z,)$. It is clear that G and F, being Hom's, are left exact.

If E is an injective $\prod-$module, we claim $GE = E^N$ is $\prod/N$-injective (hence right F-acyclic). Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & E^N \arrow[r] \arrow[d, "f"'] \arrow[d, phantom, "t", shift right=8ex] & E \arrow[d] \\
0 \arrow[r] & M' \arrow[r, "i"'] & M
\end{tikzcd}$$

Where f and i are $\prod N-$maps. By change of rings $Z\prod \to Z(\prod/N)$, every $\prod/N-$module may be regarded as a $\prod$-module and every $\prod/N-$map may be regarded as a $\prod$-map. Since E is $\prod$-injective, there is a $\prod$-map $\tilde{f}: M \to E$ extending f. But im $\tilde{f} \subset E^N : n.\tilde{f}(x) = \tilde{f}(n.x) = \tilde{f}(x)$, since each $x \in M$ is fixed by every $n \in N$. It follows that $\tilde{f}: M \to E^N$ is a $\prod/N$-map, and E^N is $\prod/N$-injective.

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