*International Journal of Engineering, Science and Mathematics*
Vol. 6 Issue 7, November 2017,
ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijesm.co.in, Email: ijesmj@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gate as well as in Cabell's Directories of Publishing Opportunities, U.S.A

# RING-THEORETIC STRUCTURES WHICH DO NOT INCLUDE "CLOSEDNESS" CONSTRAINTS

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***Abstract.*** In this paper, we analyze the ring $\nabla[\mathcal{L}^4, \mathcal{L}^5] = \nabla + \mathcal{L}^4\nabla[\mathcal{L}]$ and establish some parallel results to the polynomial ring $\nabla[\mathcal{L}]$. Our main objective is to identify that $\nabla[\mathcal{L}^4, \mathcal{L}^5]$ is a UMT domain except if $\nabla$ is a UMT domain.

***Keywords.*** Eulerian property, integral domain, ideal, UMT domain.

## 1. *Introduction*

Consider $\nabla$ as an integral domain, let $\mathcal{L}$ be an indeterminate domain over $\nabla$, and let $\nabla[\mathcal{L}]$ be a polynomial ring over $\nabla$. A nonzero prime ideal $\nabla'$ of $\nabla$ $[\mathcal{L}]$ is referred to as upper to zero in $\nabla[\mathcal{L}]$ if $\nabla'$ of $\nabla = (0)$ is defined. We state that $\nabla$ is a UMT-domain if each upper to zero in $\nabla[\mathcal{L}]$ is the maximum t-ideal of $\nabla[\mathcal{L}]$. Houston and Zafrullah presented the notion of UMT-domains in 1989.

Throughout the whole manuscript, $\nabla$ represents an integral domain with quotient field $\nabla'$. For $\varphi \in \nabla''[\mathcal{L}]$, also considering $\mathcal{A}_\varphi$ is the fractional ideal of $\nabla$ which is generated by the coefficientsof $\varphi$. The idea of multiplicative ideal theory is better explained by Gilmer [2] whereas the commutative ring theory is discussed in detail by Kaplansky [8]. The important Lemmas to support our focal results are proved to make the manuscript self-contained and are as follows :

**Lemma 2.1** Let $\mathfrak{l}$ be a nonzero fractional ideal of $\nabla$. Then

(a) $(\mathfrak{l}\,\nabla[\mathcal{L}^4,\mathcal{L}^5])^{-1} = (\mathfrak{l})^{-1}\nabla[\mathcal{L}^4,\mathcal{L}^5]$.

(b) $(\mathfrak{l}\,\nabla[\mathcal{L}^4,\mathcal{L}^5])_\vartheta = (\mathfrak{l})_\vartheta\nabla[\mathcal{L}^4,\mathcal{L}^5]$.

(c) $(\mathfrak{l}\,\nabla[\mathcal{L}^4,\mathcal{L}^5])_\tau = (\mathfrak{l})_\tau\nabla[\mathcal{L}^4,\mathcal{L}^5]$.

***Proof.***(a) It is obvious that $(\mathfrak{l})^{-1}\nabla[\mathcal{L}^4,\mathcal{L}^5]$ is an subset of $(\mathfrak{l}\,\nabla[\mathcal{L}^4,\mathcal{L}^5])^{-1}$. It is important to note that $\mathfrak{l}(\mathfrak{l}\,\nabla[\mathcal{L}^4,\mathcal{L}^5])^{-1}$ is subset of $\nabla[\mathcal{L}^4,\mathcal{L}^5]$is subset of$\nabla'[\mathcal{L}^4,\mathcal{L}^5]$, it results
$(\mathfrak{l}\,\nabla[\mathcal{L}^4,\mathcal{L}^5])^{-1}$ is subset of $\nabla[\mathcal{L}^4,\mathcal{L}^5]$.

If $\varphi$ belongs to $(\mathfrak{l}\,\nabla[\mathcal{L}^4,\mathcal{L}^5])^{-1}$, then $\mathcal{A}_\varphi\mathfrak{l}$ is a subset of $\nabla$. And hence we have, $\varphi \in$ $\mathcal{A}\varphi\nabla\mathcal{L}4,\mathcal{L}5$ is a subset of $(\mathfrak{l})-1\nabla\mathcal{L}4,\mathcal{L}5$. Results,

$$(\mathfrak{l}\,\nabla[\mathcal{L}^4,\mathcal{L}^5])^{-1} = (\mathfrak{l})^{-1}\nabla[\mathcal{L}^4,\mathcal{L}^5].$$

(b)Using (a),

$$(\mathfrak{l}\,\nabla[\mathcal{L}^4,\mathcal{L}^5])_\vartheta = ((\mathfrak{l}\nabla[\mathcal{L}^4,\mathcal{L}^5])^{-1})^{-1}$$
$$= ((\mathfrak{l})^{-1}\nabla[\mathcal{L}^4,\mathcal{L}^5])^{-1}$$
$$= (\mathfrak{l})_\vartheta\nabla[\mathcal{L}^4,\mathcal{L}^5].$$

(c) it is obvious that if $(\varphi)_1, (\varphi)_2, (\varphi)_3, (\varphi)_4 \ldots (\varphi)_k \in \mathfrak{l}\,\nabla[\mathcal{L}^4,\mathcal{L}^5]$ and hence we have

$$((\varphi)_1, (\varphi)_2, (\varphi)_3, (\varphi)_4 \ldots (\varphi)_k)$$
$$\subseteq ((\mathcal{A}(\varphi)_1, \mathcal{A}(\varphi)_2, \mathcal{A}(\varphi)_3, \mathcal{A}(\varphi)_4 \ldots \mathcal{A}(\varphi)_k)\nabla[\mathcal{L}^4,\mathcal{L}^5])_\vartheta$$
$$= ((\mathcal{A}(\varphi)_1, \mathcal{A}(\varphi)_2, \mathcal{A}(\varphi)_3, \mathcal{A}(\varphi)_4 \ldots \mathcal{A}(\varphi)_k))_\vartheta\nabla[\mathcal{L}^4,\mathcal{L}^5]$$
$$\subseteq (\mathfrak{l})_\tau\nabla[\mathcal{L}^4,\mathcal{L}^5]$$

and hence
$(\mathfrak{l}\,\nabla[\mathcal{L}^4,\mathcal{L}^5])_\tau$ is a subset of $(\mathfrak{l})_\tau\nabla[\mathcal{L}^4,\mathcal{L}^5]$. Contrarily, let nonempty $\Gamma$ be a finitely generated subideal of $\mathfrak{l}$. Then $(\Gamma)_\vartheta\nabla[\mathcal{L}^4,\mathcal{L}^5] = (\Gamma\nabla[\mathcal{L}^4,\mathcal{L}^5])_\vartheta$ is a subset of $(\mathfrak{l}\nabla[\mathcal{L}^4,\mathcal{L}^5])_\tau$ by (b). Which results $(\mathfrak{l})_\tau\nabla[\mathcal{L}^4,\mathcal{L}^5]$ is a subset of $(\Gamma\nabla[\mathcal{L}^4,\mathcal{L}^5])_\tau = (\Gamma)_\tau\,\nabla[\mathcal{L}^4,\mathcal{L}^5]$.

**LEMMA 2.** Let $\nabla'$ be a maximal $t-ideal$ of $\nabla[\mathcal{L}^4,\mathcal{L}^5]$ such that $\nabla'\cap\nabla$ is nonempty. Then $\nabla' = (\nabla'\cap\nabla)[\mathcal{L}^4,\mathcal{L}^5]$. Particularly, $\nabla'\cap\nabla$ is a maximal $t-ideal$ of $\nabla$.

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***Proof.*** It is sufficient to prove that

$c(\nabla')[\mathcal{L}^4,\mathcal{L}^5]$ is a subset of $\nabla'$. $\nabla'$ generated an ideal $c(\nabla')$ with the help of coefficients of all the polynomials in $\nabla'$. If $c(\nabla')$ is not a subset of $\nabla'$, which gives

$\nabla'$is a subset of $c(\nabla')[\mathcal{L}^4,\mathcal{L}^5]$.

For $\nabla'$ Being a maximal $t-ideal$, we have

$$\left(c(\nabla')\right)_\tau[\mathcal{L}^4,\mathcal{L}^5] = (c(\nabla')[\mathcal{L}^4,\mathcal{L}^5])_\tau$$
$$= \nabla[\mathcal{L}^4,\mathcal{L}^5].$$

It results,

$\left(c(\nabla')\right)_\tau = \nabla;$

Whenever there is an existence of a polynomial $\varphi$ belongs to $\nabla'$ in such a way that

$(\mathcal{A}_\varphi)_\vartheta = \nabla.$

Consider a nonempty $\alpha$ is an element of $\nabla' \cap \nabla$.

Claiming the condition $(\alpha,\varphi)^{-1} = \nabla[\mathcal{L}^4,\mathcal{L}^5]$. Since $\gamma$ is an arbitrarily chosen of $(\alpha,\varphi)^{-1}$ and $\alpha\gamma \in \nabla'[\mathcal{L}^4,\mathcal{L}^5]$, it is to be noted that $(\alpha,\varphi)^{-1}$ is a subset of $\nabla'[\mathcal{L}^4,\mathcal{L}^5]$. Also, if $\in (\alpha,\varphi)^{-1}$, there is an existence of an integer $\mu \geq 1$ in such a way $\mathcal{A}_\varphi^{\mu+1}\mathcal{A}_{\varphi\gamma}$ [2]. And hence $(\mathcal{A}_\varphi^{\mu}\mathcal{A}_{\varphi\gamma})_\vartheta = ((\mathcal{A}_\varphi^{\mu})_\vartheta\mathcal{A}_{\varphi\gamma})_\vartheta = (\mathcal{A}_{\varphi\gamma})_\vartheta$ is a subset of $\nabla'$.

Results,

$\gamma \in \mathcal{A}_\varphi[\mathcal{L}^4,\mathcal{L}^5]$ is a subset of $\nabla[\mathcal{L}^4,\mathcal{L}^5]$,

Hence,

$(\alpha,\varphi)^{-1} = \nabla[\mathcal{L}^4,\mathcal{L}^5]$, which gives $(\alpha,\varphi)_\vartheta = \nabla[\mathcal{L}^4,\mathcal{L}^5]$, which is a contradictory statement because $\nabla'$ is a $t-ideal$. Now, we have

$$c(\nabla')[\mathcal{L}^4,\mathcal{L}^5] = \nabla'$$

And hence finally we have

$$\nabla' = (\nabla \cap \nabla')[\mathcal{L}^4,\mathcal{L}^5].$$

As in [1], $\nabla$ is referred to as the UMT-domain if it is any upper to zero (nonzero prime). The ideal of $\nabla[\mathcal{L}]$ which contracting to zero in is the maximum $t-ideal$. Recollect that $\nabla[\mathcal{L}]$ is a UMT-domain if and only if $\nabla$ is a UMT-domain [6, Theorem 3.4]. As just a consequence of our next result, $(\nabla)[\mathcal{L}^4,\mathcal{L}^5]$ is a UMT-domain if and only If $\nabla[\mathcal{L}]$ is a UMT domain.

**Theorem.** $(\nabla)[\mathcal{L}^4,\mathcal{L}^5]$ is a UMT iff $\nabla$ is a UMT-domain.

***Proof.***Considering $(\nabla)[\mathcal{L}^4,\mathcal{L}^5]$ as a UMT-domain. Assuming a maximal $t-ideal\beta$ of $\nabla$. Then by using Lemma [2], a maximal $t-ideal$ $(\beta\nabla)[\mathcal{L}^4,\mathcal{L}^5]$of $(\nabla)[\mathcal{L}^4,\mathcal{L}^5]$. Also, it is to be noted that $(\nabla)[\mathcal{L}^4,\mathcal{L}^5]_{\beta\vartheta(\nabla)[\mathcal{L}^4,\mathcal{L}^5]} = (\nabla)[\mathcal{L}]_{\beta[\mathcal{L}]}$. It is given that, $(\nabla)[\mathcal{L}^4,\mathcal{L}^5]$ is a UMT-domain, $\nabla[\mathcal{L}]_{\beta[\mathcal{L}]}$ is a $t$ −linkative UMT-domain [6], which gives $\nabla_\beta$ is a $t$ −linkative UMT-domain. Which results that $\nabla$ is a UMT-domain.

Converse,

Considering a UMT-domain $\nabla$. It is to prove that $\nabla[\mathcal{L}^4,\mathcal{L}^5]$ is a UMT-domain. Now it will be more than sufficient to show that $\nabla'$ is a maximal ideal of $\nabla[\mathcal{L}^4,\mathcal{L}^5]$, then there is an existence of prufer domain of integral closure of $(\nabla[\mathcal{L}^4,\mathcal{L}^5])_{\nabla'}$.

Considering a maximal $t-ideal\nabla'$ of $\nabla[\mathcal{L}^4,\mathcal{L}^5]$ and let $\nabla' \cap \nabla[\mathcal{L}^4,\mathcal{L}^5] = \beta$. For nonempty $\beta$, $\nabla' = \beta[\mathcal{L}^4,\mathcal{L}^5]$. Additionally, since $\mathcal{L}^4$ doesn't belongs to $\beta[\mathcal{L}^4,\mathcal{L}^5]$ we have $\nabla[\mathcal{L}^4,\mathcal{L}^5]_{\nabla'} = \nabla[\mathcal{L}]_{\beta[\mathcal{L}]}$.

Thus we have a Prufer domain which is the integral closure of $\nabla[\mathcal{L}^4,\mathcal{L}^5]_{\nabla'}$. For zero $\beta$, then $\nabla[\mathcal{L}^4,\mathcal{L}^5]_{\nabla'} = \nabla''[\mathcal{L}^4,\mathcal{L}^5]_{\nabla'\nabla''[\mathcal{L}^4,\mathcal{L}^5]}$, which shows that $\nabla[\mathcal{L}^4,\mathcal{L}^5]_{\nabla'}$ is uni-dimensional Noetherian domain which results that we have Dedekind domain which is nothing but the integral closure of $\nabla[\mathcal{L}^4,\mathcal{L}^5]_{\nabla'}$ and hence Prufer domain.

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