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Metric Fixed Point Theory- The Role of Real Line in Calculus

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1. Metric Fixed Point Theory

Metric spaces are elementary in functional analysis because they play a role similar to the role of real line in calculus. In fact, they generalize real line and provide a basis for a unified treatment of significant problems in mathematics and applied sciences. In 1906, Maurice Frechet, a French mathematician, presented an idea of metric spaces. The word metric is derived from the word metor which means measure. Further, he pioneered the study of these spaces and their applications to different areas of mathematics. In the past three decades, metric spaces have gained much attention due to the advancement of metric fixed point theory.

Definition 1.1 (1) A real valued function $\rho : Z \times Z \to Z$ where Z is a nonempty setiscalled a distance function or metric in Z, the following conditions are satisfied:

1. $\rho(u, v) \geq 0$
2. $\rho(u, v) = \rho(v, u)$,(Symmetry)
3. $\rho(u, v)$=0ifandonlyif$u$=v,(Positive Definiteness)
4. $\rho(u, v) \leq \rho(u, z) + \rho(z, v)$.(Triangle Inequality) Then the ordered pair(Z, ρ) is **Metric Space**.

Definition 1.2. (2) A function $Q : [a, b] \to R$ is called **bounded** if ∃ a real number$k$>0 such that

$$|Q(u)| \leq k, \qquad \forall u \in [a, b].$$

Example 1.1. (2) If X = Body Space ($u \in R^3$: co-ordinates points implied by a card-aver frozen in R^3). Define $\rho(u, v)$ to be Euclidean length of the shortest path lying entirely within X which connects u and v. This is a metric and the distance from a toe nail to a finger tip does not depend on the configuration of the body, where as the usual spatial distance would. This metric is very useful in anatomy.

Ancommonexamplewhichshowsthateverydistancefunctioninarealspaceis not a metric space, is given below.

Example 1.2. (2) A real valued function $\rho : R \times R \to R$ is given by

$$\rho(u, v) = u^2 - v^2, \qquad \forall u, v \in Z.$$

Then the ordered pair (Z, ρ) is not a metric space because the property (1)

Definition 1.3. (2) A sequence $\{u_n\}$ of points in a metric space (Z, ρ) is said to be **convergent sequence** if ∃ a point $z \in Z$ *s.t.*

$$\lim_{n\to\infty} \rho(u_n, z) = 0.$$

Remark 1.1. It is important to note that if a sequence convergent to a point then the point of convergence is unique.

Let us consider an example of a convergent sequence in metric spaces.

Example 1.3. (2) Let $Z = R$. Define a function $\rho : Z \times Z \to R^+$ by

$$\rho(u, v) = |u - v|, \qquad \forall u, v \in Z,$$

Definition 1.2.4. (2) A sequence $\{u_n\}$ of points in a metric space (Z, ρ) is called **Cauchy sequence** if, $\forall \varepsilon > 0$, $\exists +v\ e$ integer n_0 *s.t.*

$$\rho(u_n, u_m) < \varepsilon, \qquad \forall n, m > n_0.$$

Theorem 1.1. (2) Every convergent sequence in a metric space is a Cauchy se- quence but converse need not be true.

Definition1.4. (2) A metric space (Z, ρ) is called a complete metric space, if every Cauchy sequence in Z converges to a point in Z.

Remark 1.2. (2) A complete subspace is the closed subset of a complete metric space .

Definition 1.6. (2) Let us consider a metric space (Z, ρ) and A is a nonempty subset, then the diameter of A is defined as

$$diam(A) = \sup\{\rho(u, v) | u, v \in A\} .$$

Then A is said to be bounded if diam(A) is finite.

Example 1.2.4. (2) Let $Z_1 = R^2$ be associated with the metric d_1 given by

$$\rho_1(u,v)=\rho_1((u_1,u_2),(y_1,y_2))=\sqrt{(u1-y1)^2+(u2-y2)^2}$$

for all $u = (u_1, u_2)$, $v = (y_1, y_2) \in Z_1$ and $Z_2 = R$ be equipped with usual metric ρ_2. The function $Q :(Z_1,\rho_1) \to (Z_2, \rho_2)$ defined by $Q (u, v) = u + v$ for each $u,v \in Z_1$,is continuous.

Definition 1.7. (3) A mapping $Q :Z \to Z$ on (Z, ρ) is said to fulfill the Lipschitz condition, if ∃ a real number k such that

$$\rho (Qu, Qv) \leq k\rho(u,v), \qquad \forall u, v \in Z.$$

1.2 Contraction Mappings and Their Generaliza- tions

Metric fixed point theory has its heredity in strategies since the start of nineteenth century, when Cauchy proposed successive approximations methods to found the existence and uniqueness of solutions to various types of equations, and particularly differential equations. In 1912, Picard proposed the second method of successive approximations for proving the existence theorems. It is also known as Picard's method. Although, In1922 Stefan Banach (4) established the thoughts required in conceptual setting and proved fixed point theorem, generally known as the Banach contraction principle.

Definition 1.2.1. (4) Let us consider a metric space (Z, ρ) and a mapping $Q: Z \to Z$ is said to be a contraction mapping, if ∃ real number $k \in [0, 1)$ *s.t.*

$$\rho (Qu, Qv) \leq k\rho(u,v), \qquad \forall u, v \in Z,$$

Example 1.2.1. (5) If Q be a mapping defined as $Q(u) = \frac{u}{t}$, where t is a real number greater than one. Then Q is a contraction mapping.

Remark 1.2.1. (5) Every contraction mapping is clearly continuous but conversely need not be true. Also, if the space Z is not complete, then it does not guarantee the fixed point of a contraction map.

For example:

Example 1.2.2. (5)Consider a mapping $Q:(0, 1] \to (0, 1]$ defined by $Q(u) = \frac{2x}{5}$. Although it is a contraction mapping but it has no fixed point as the space (0, 1] is not a complete metric space.

Remark 1.2.2. (5) If $k = 1$ in Definition 1.2.7, then Q is called a non-expansive map- ping. Moreover, if $\rho (Qu, Qv) < \rho(u, v)$, $u \neq v$ then Q is said to be contractive map. The non expansive mappings in fixed point theory of is not quit same as contraction mappings and the study of these mappings has been one of the main research areas of nonlinear functional analysis.

From above one can say that

Contraction ⇒Contractive ⇒Nonexpansive⇒Lipschitz.

But converse may not be true in either cases. Consider the following examples:

*Example*1.2.3. (5) Let us consider a metric space (Z, ρ).Then the identity map

$I : Z \to Z$ is non-expansive but not contractive because for $u, v \in Z$.

$$\rho(Iu, Iv) = \rho(u, v).$$

Example 1.2.4. (5) Let $Z = [1, \infty)$ and $Q : Z \to Z$ is given by $Q(u) = u + \frac{1}{u}$ Then clearly Q is contractive but not a contraction mapping.

The conclusion from above is the contraction mapping led the foundation for several mathematicians to study the problems of fixed point. Banach fixed point theorem (4) is one of the pivotal and famous result in the history of fixed point of theory.

Theorem 1.2.1. *(4) Let us consider a metric space (Z, ρ) which is complete and Qis a contraction map which fulfill*

$$\rho(Qu, Qv) \le k\rho(u, v), \ \forall u, v \in Z, (1.2.1)$$

where $k \in [0, 1)$. Then a unique fixed point exist for Q in Z. Moreover, $\forall z \in Z$, $z=Qz \forall u \in Z$, the Picard sequence $\{Q^n(u)\}$ converges to z and in fact for each $u \in Z$

$$\rho(Q^n u, z) \le \frac{k^n}{1-k}\rho(u, Qu) \text{ and } n \ge 1. (1.2.2)$$

This result of Banach has been extended, generalized and improved by several researchers. It is the easiest and one of the most flexible results in fixed point theory. Being based on an iterative scheme, it can be implemented on a computer to find the fixed point of a contractive map. It produces approximations of any required accuracy, and moreover, the number of iteration schemes needed to get a specified accuracy can be determined. It also become a very famous technique in solving existence problems in many area of mathematics.

In 1932, Banach result is applied on the proof of the existence theorem given by Caccioppoli (6).

Theorem 1.2.2. *(6) Let us consider Q is a self map defined on complete metricspace* (Z, ρ) *and fulfill the following:*

(i) $\rho(Qu, Qv) \leq ||Q||\rho(u, v)$,
(ii) $Q^n(u)=Q(Q^{n-1}(u))$,
(iii) $Q^n(v) = Q(Q^{n-1}(v))$.

Then the sequence $\{Q^n(z)\}$ *converges to a fixed point* $z_0 \in Z$ *and* $z_0 = fz_0$*, if* $\sum_{n=1}^{\infty} ||Q^n|| < \infty$*, where*

$$\rho(Q^n u, Q^n v) \leq ||Q^n||\rho(u, v).$$

Rakotch (7), in 1962, established the given result.

Theorem 1.2.3. *(7) Let* $Q:Z \to Z$ *is a mapping from complete metric space*(Z,ρ)*into it self such that*

$$\rho(Qu,Qv) \leq \lambda(\rho(u,v))\rho(u,v), \quad \forall u, v \in Z \quad \text{and} \quad \lambda \in \Lambda.$$

then Qz= z in Z where z is unique.

In 1962, Edelstein (8) established the given theorem.

Theorem 1.2.4. *(8) Let us consider a metric space* (Z, ρ) *which is complete and a map* $Q : Z \to Z$ *such that*

$$\rho (Qu, Qv) < \rho(u,v), \quad \forall \quad u, v \in Z, \quad u \neq v,$$

then Qz= z in Z where z is unique.

In 1966, Sehgal (9) established the given result.

Theorem 1.2.5. *(9) Let us consider a metric space* (Z, ρ) *which is complete and* $Q : Z \to Z$ *is a continuous mapping satisfying the condition that* $\exists$*a number* $k < 1$ *s.t for each* $u \in Z$*, there is positive integer* $n = n(u)$ *such that*

$$\rho(Q^n u, Q^n v) \leq k\rho(u, v), \ \forall u \in Z. (1.3.3)$$

then Qz= z in Z where z is unique.

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