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The Killing Vectors for Spherically Symmetric Space-time in Wide Sense

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Abstract

In this paper the Killing vectors admitted by a spherically symmetric Space-time using a Spherically Symmetric co-ordinate system in wide sense is studied.

Keywords: Killing vectors; Spherically symmetric.

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1 Introduction

The concept of spherical symmetry is connected with the group of motion which satisfies the Killing equation

$$K_{i;j} + K_{j;i} = 0 \quad (1)$$

(; represent the covariant derivative.)
The vector K^i is called the Killing vector. As it plays a major role in spherical symmetry, it is desirable to identify it.

In this paper we have taken up the most general spherically symmetric line ele ment

$$ds^2 = -Adr^2 - B(d\theta^2 + \sin^2\theta d\varphi^2) + Cdt^2 + 4Ddrdt, \quad (2)$$

where A, B, C and D are the functions of r and t, for the determination of the Killing vectors.

2 Killing equation

The Killing equation (1) can be written as

$$K^h g_{ij,h} + g_{ih} K^h_{,j} + g_{hj} K^h_{,i} = 0$$

(, denotes the partial derivative) which for the line element (2) gives

$$A' K^1 + 2A K^1_{,r} + \dot{A} K^4 - 4D K^4_{,r} = 0, \quad (3)$$

$$A K^1 + B K^2 - 2D K^4 = 0, \quad (4)$$

$$A K^1 \quad + B \quad \sin^2 \quad = 0, \quad (5)$$
$$\theta K^3 - 2D K^4$$

$$2D' K^1 - A K^1 + 2D K^1 + 2\dot{D} \quad = 0, \quad (6)$$
$$\mathbf{M}$$
$$K^4 + 2D K^4 + C K^4$$

$$B' K^1 + 2B K^2 + \dot{B} \ \mathrm{M} \ K^4 = 0, \quad (7)$$

$$^2 + \sin^2 \theta K^3 \quad = 0, \quad (8)$$

$$2D K^1 - B K^2 + \quad = 0, \quad (9)$$
$$C K^4$$

$$B' \sin\theta K^1 + 2B \cos\theta K^2 + 2B \sin\theta K^3 + \dot{B} \ \mathbf{M} \ \sin\theta K^4 = 0, \quad (10)$$

$$2D K^1 \quad = 0,$$
$$-B \sin\theta K + CK \quad (11)$$

$$C' K^1 + 4D K^1_{,t} + C \ \mathrm{M} \ K^4 + 2C K^4_{,t} = 0, \quad (12)$$

where $\dot{A} = A_{,t}$ and $A' = A_{,r}$ etc.

These are ten equations in four K^i, $i = 1, 2, 3, 4$ which are the functions of (r, θ, φ, t).

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Differentiating equation (4) and (5) with respect to φ and θ respectively, we obtain

$$-AK^1{}_{,\theta\varphi} - BK^2{}_{,r\varphi} + 2DK^4{}_{,\theta\varphi} = 0,$$

and

$$-AK^1{}_{,\theta\varphi} - 2B\sin\theta\cos\theta K^3{}_{,r} - B\sin^2\theta K^3{}_{,r\theta} + 4DK^4{}_{\theta\varphi} = 0.$$

Their K^1 and K^4 eliminant yield

$$-K^2{}_{,r\varphi} + 2\sin\theta\cos\theta K^3{}_{,r} + \sin^2\theta K^3{}_{,r\theta} = 0.$$

Then using equation (8) we get

$$+ \cot\theta K^3 \quad = 0,$$

which on integration gives

$$K^3 + \cot\theta K^3 = -a_1(\varphi, t). \qquad (13)$$

This a_1 should be a function of (θ, φ, t) but to avoid the tedious nature of the results, we assume $a_1 = a_1(\varphi, t)$. [Takeno(1966) has also adopted this view].
Equation (13) is the linear partial differential equation in K^3 with integrating factor $\sin\theta$. Therefore its solution is

$$K^3 = a_1(\varphi, t)\cot\theta + a_2(r, \varphi, t)/\sin\theta. \qquad (14)$$

Taking the help of equation (7), equation (10) simplifies to

$$-\sin\theta K_{,\theta} + \cos\theta K + \sin\theta K^3 = 0.$$

Differentiating this partially with respect to φ and using equation (8), we get

$$\sin^2\theta K^3{}_{,\theta\theta} + \sin\theta\cos\theta K^3{}_{,\theta} + K^3{}_{,\varphi\varphi} = 0.$$

Substituting the value of K^3 from (14) we obtain

$$a_1 = b_1\cos\varphi + b_2\sin\varphi,\ b = b(t)$$

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$$a_2 = d_1 \cos\varphi + d_2 \sin\varphi,\ d = d(r, t).$$

Finally we get

$$K^3 = (b_1 \cos\varphi + b_2 \sin\varphi) \cot\theta + (d_1 \cos\varphi + d_2 \sin\varphi)/ \sin\theta \qquad (15)$$

Now equation (8), when combined with equation (15), gives K^2 as

$$K^2 = (b_1 \sin\varphi - b_2 \cos\varphi) + (d_1 \sin\varphi - d_2 \cos\varphi) \cos\theta + g, \qquad (16)$$

where $g = g(r, \theta, t)$.

Now equations (7), (8), (10) and (16) give

$$g_{,\theta\theta} - \cot\theta g_{,\theta} + cosec^2\theta g = 0,$$

and one of its solution is

$$g = -\sin\theta d_3(r, t).$$

Then (16) simplifies to

$$K^2 = (b_1 \sin\varphi - b_2 \cos\varphi) + (d_1 \sin\varphi - d_2 \cos\varphi) \cos\theta - d_3 \sin\theta. \qquad (17)$$

Differentiating equation (7) with respect to θ and then using equation (4), we obtain

$$(2DB' + A\dot{B})K^1 + B\dot{B}K^2 + 4BDK^2 = 0$$

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With the help of (17) above equation becomes

$$K^1_{,\theta} = \frac{1}{(2DB' + A\dot{B})} \times$$

$$\{[(4BDd_1 - B\dot{B}d_{1,r}) \sin\phi - (4BDd_2 - B\dot{B}d_{2,r}) \cos\phi] \cos\theta - (4BDd_3 - B\dot{B}d_{3,r}) \sin\theta\}$$

i.e.

$$K^1 = \frac{1}{(2DB' + A\dot{B})} \times$$

$$\{[(4BDd_1 - B\dot{B}d_{1,r}) \sin\phi - (4BDd_2 - B\dot{B}d_{2,r}) \cos\phi] \sin\theta + (4BDd_3 - B\dot{B}d_{3,r}) \cos\theta\} + f, \tag{18}$$

Where f = f(r, ϕ ,t)

We obtain as follows

Equations (5) and (7) give

$$-B\dot{B}\sin^2\theta K^3_{,r} - 4BDK^2_{,\theta\phi} - (A\dot{B} + 2DB')K^1_{,\phi} = 0,$$

and then putting the values of K^1, K^2 and K^3 this equation yields

$$f_{,\phi} = 0 \quad i.e. \quad f = f(r,t)$$

Equation (7) with the help of equation (17) and (18) becomes

$$K^4 = \frac{B}{2DB' + A\dot{B}} \times$$

$$\{[(B'd_{1,r} + 2Ad_1) \sin\phi - (B'd_{2,r} + 2Ad_2) \cos\phi] \sin\theta + (B'd_{3,r} + 2Ad_3) \cos\theta\} - \frac{B'}{\dot{B}} f.$$

Substituing the expression for K^1, K^2, K^3 and K^4 in equations (3) and (12) we write

$$-A'f + \frac{B'\dot{A}f}{\dot{B}} - 2Af_{,r} + 4D\left(\frac{-B'f}{\dot{B}}\right)_{,r} = 0, \tag{19}$$

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$$C'f - \frac{B'\dot{C}f}{\dot{B}} + 4Df_{,t} + 2C\left(\frac{-B'f}{\dot{B}}\right)_{,t} = 0. \tag{20}$$

Equation (6) with equation (19) and (20) gives $Xf = 0$, where

$$X = 2D' - \frac{2B'\dot{D}}{\dot{B}} + \frac{CA'}{4D} - \frac{CB'\dot{A}}{4D\dot{B}} - \frac{DC'}{C} + \frac{DB'\dot{C}'}{C\dot{B}} +$$
$$\frac{(2D + \frac{2AC}{4D})}{(2A + \frac{4DB'}{\dot{B}})}\left[-A' + \frac{B'\dot{A}}{\dot{B}} + \frac{4D(-\dot{B}B'' + B'\dot{B}')}{\dot{B}^2}\right] +$$
$$\frac{(-A - \frac{4D^2}{C})}{(-4D + \frac{2CB'}{\dot{B}})}\left[C' - \frac{B'\dot{C}}{\dot{B}} + \frac{2C(-\dot{B}\dot{B}' + B'\ddot{B})}{\dot{B}^2}\right],$$

Then either $X = 0$ or $f = 0$.

For $f = 0$

$$K^1 = \frac{1}{(2DB' + A\dot{B})}[(4BDd_1 - B\dot{B}d_{1,r})\sin\phi - (4BDd_2 - B\dot{B}d_{2,r})\cos\phi]\sin\theta +$$
$$\frac{1}{(2DB' + A\dot{B})}(4BDd_3 - B\dot{B}d_{3,r})\cos\theta, \tag{21}$$

and

$$K^4 = \frac{B}{(2DB' + A\dot{B})}[(B'd_{1,r} + 2Ad_1)\sin\phi - (B'd_{2,r} + 2Ad_2)\cos\phi]\sin\theta +$$
$$\frac{B}{(2DB' + A\dot{B})}(B'd_{3,r} + 2Ad_3)\cos\theta. \tag{22}$$

Then expressions (15), (17), (21) and (22) completely determine the Killing vector K^i.

Now we can obtain more information about the functions $d = d(r, t)$. Noting the values of K^1, K^2 and K^4 equation (9) implies an identity

$$\frac{2D(4BDd_a - B\dot{B}d_{a,r})}{(2DB' + A\dot{B})}[(\sin\phi - \cos\phi)\cos\theta - \sin\theta] +$$

$$\frac{CB(B'd_{a,r} + 2Ad_a)}{(2DB' + A\dot{B})}[(\sin\phi - \cos\phi)\cos\theta - \sin\theta] -$$

$$[b_{1,t}\sin\phi - b_{2,t}\cos\phi + (d_{1,t}\sin\phi - d_{2,t}\cos\phi)\cos\theta - d_{3,t}\sin\theta] = 0.$$

Equating the terms of $\sin\phi\cos\theta, \cos\phi\cos\theta, \sin\theta, \sin\phi$ and $\cos\phi$ we get

$$(BD^2 + 2AC)d_a + (-2\dot{B}D + B'C)d_{a,r} + (-2B'D - A\dot{B})d_{a,t} = 0 \qquad (23)$$

for $a = 1, 2, 3$ and

$$b_{1,t} = b_{2,t} = 0 \qquad (24)$$

or $B = 0$, but $B \neq 0$. Therefore (23) and (24) are the only possibilities. Hence d_a satisfies (23).

3 Isotropic co-ordinate system

In isotropic coordinate system the line element is written as

$$ds^2 = -A[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] + Cdt^2.$$

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This is obtained from (2) for $D = 0$ and $B = Ar^2$, where A and C are the functions of r and t.

In this case

$$K^1 = -r^2[(d_{1,r}\sin\phi - d_{2,r}\cos\phi)\sin\theta + d_{3,r}\cos\theta].$$

Now equation (23) becomes

$$2ACd_a + (2Ar + A'r^2)Cd_{a,r} = 0$$

provided $\dot{A} = 0$. Then

$$d_{a,r} = \frac{-2Ad_a}{(2Ar + A'r^2)}, \tag{25}$$

and

$$K^1 = \frac{2Ar^2}{(2Ar + A'r^2)}[(d_1\sin\phi - d_2\cos\phi)\sin\theta + d_3\cos\theta].$$

with these, equation (3) becomes

$$2AA' + 3(A')^2 r - 2AA''r = 0, \quad i.e. \quad \frac{r^2A^3}{(A')^2} = c^2(constant)$$

or

$$A = \frac{1}{(e_1r^2 + e_2)^2},$$

where e_1 and e_2 are arbitrary constants. Now equation (25) becomes

$$d_a = \left[\frac{(e_1r^2 - e_2)}{(e_1r)}\right]p_a, \quad p_a = p_a(t).$$

The quantity C is obtained from equation (6) and (9) as

$$C = \frac{(e_1r^2 - e_2)^2 q}{(e_1r^2 + e_2)^2}$$

where $q = q(t)$.

By a suitable transformation of t, we may have $q(t) = 1$. It is interesting to note that K^4 can not be determined from (22) because its denominator becomes zero as $\dot{A} = 0$.

However K^4 can be determined as follows. Equation (9) gives

$$K^4_{,\theta} = \frac{r^2(e_1r^2 + e_2)^2}{(e_1r^2 + e_2)^2(e_1r^2 - e_2)^2}[(d_{1,t}\sin\phi - d_{2,t}\cos\phi)\cos\theta + d_{3,t}\sin\theta],$$

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or

$$K^4 = \left[\frac{r}{e_1(e_1r^2 - e_2)}\right][(\dot{p_1}\sin\phi - \dot{p_2}\cos\phi)\sin\theta + \dot{p_3}\cos\theta].$$

To determine function $p's$ we consider equation (12)

$$C'K^1 + 2CK^4_{,t} = 0,$$

$$i.e.\quad \ddot{p_a} - 4e_1e_2p_a = 0.$$

or

$$p_a = k_{1a}e^{2\sqrt{e_1e_2}\,t} + k_{2a}e^{-2\sqrt{e_1e_2}\,t},$$

where k_{1a} and k_{2a} are arbitrary constants.

Thus the componants of Killing vector in this case are given by

$$K^1 = -\frac{(e_1r^2 + e_2)}{e_1}\times$$

$$\{[(k_{11}e^{mt}+k_{21}e^{-mt})\sin\phi-(k_{12}e^{mt}+k_{22}e^{-mt})\cos\phi]\sin\theta+(k_{13}e^{mt}+k_{23}e^{-mt})\cos\theta\},$$

$$K^2 = (b_1\sin\phi - b_2\cos\phi) + \frac{(e_1r^2 - e_2)}{e_1r}\times$$

$$\{[(k_{11}e^{mt}+k_{21}e^{-mt})\sin\phi-(k_{12}e^{mt}+k_{22}e^{-mt})\cos\phi]\cos\theta-(k_{13}e^{mt}+k_{23}e^{-mt})\sin\theta\},$$

$$K^3 = (b_1\cos\phi + b_2\sin\phi)\cot\theta + \frac{(e_1r^2 - e_2)}{e_1r}\times$$

$$\{[(k_{11}e^{mt} + k_{21}e^{-mt})\cos\phi + (k_{12}e^{mt} + k_{22}e^{-mt})\sin\phi]\}/\sin\theta,$$

$$K^4 = \frac{rm}{e_1(e_1r^2 - e_2)}\times$$

$$\{[(k_{11}e^{mt}-k_{21}e^{-mt})\sin\phi-(k_{12}e^{mt}-k_{22}e^{-mt})\cos\phi]\sin\theta+(k_{13}e^{mt}-k_{23}e^{-mt})\cos\theta\},$$

where $m = 2\sqrt{e_1e_2}$

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